Algorithms for the Padé approximants used in HEPA Computation

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1 Padé approximant of $f(z)$

Let the function $f(z)$ be given and its Taylor expansion be:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j.$$

Let the two polynomials $P_{n,m} \in \Pi_n$, $Q_{n,m} \in \Pi_m$ have no common divisor, $Q_{n,m} \neq 0$, and satisfy the condition:

$$Q_{n,m}(z)f(z) - P_{n,m}(z) = \mathcal{O}\left(z^{n+m+1}\right).$$

We set

$$\pi_{n,m}(z) = \frac{P_{n,m}(z)}{Q_{n,m}(z)} = \frac{p_n z^n + p_{n-1}z^{n-1} + \ldots + p_0}{q_m z^m + q_{m-1}z^{m-1} + \ldots + q_0}$$

for the Padé approximant of $f(z)$. Therefore we have that:

$$\begin{aligned}(q_m z^m + q_{m-1}z^{m-1} + \ldots + q_0)(c_0 + c_1 z + \ldots + c_{n+m}z^{n+m}) - \\ -(p_n z^n + p_{n-1}z^{n-1} + \ldots + p_0) = \mathcal{O}\left(z^{n+m+1}\right).\end{aligned}$$

For example, let $n = m = 3$. Then:

$$\begin{aligned}(q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6) - \\ -(p_3 z^3 + p_2 z^2 + p_1 z + p_0) = \mathcal{O}\left(z^7\right).\end{aligned}$$

Comparing each degree, we have:

$$\begin{aligned}
z^0 &: q_0 c_0 - p_0 = 0 \\
z^1 &: q_1 c_0 + q_0 c_1 - p_1 = 0 \\
z^2 &: q_2 c_0 + q_1 c_1 + q_0 c_2 - p_2 = 0 \\
z^3 &: q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 - p_3 = 0 \\
z^4 &: q_3 c_1 + q_2 c_2 + q_1 c_3 + q_0 c_4 = 0 \\
z^5 &: q_3 c_2 + q_2 c_3 + q_1 c_4 + q_0 c_5 = 0 \\
z^6 &: q_3 c_3 + q_2 c_4 + q_1 c_5 + q_0 c_6 = 0
\end{aligned}$$

The first four equations are enough to compute the coefficients of $P_{n,m}$. The last three equations have four variables. We set $q_0 = 1$, and we write the three equations as a matrix:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} -c_4 \\ -c_5 \\ -c_6 \end{pmatrix}.$$

We solve this matrix equations and we have the coefficients of $Q_{n,m}$. Now we compute the coefficients of $P_{n,m}$ with this formula:

$$p_n = c_n + \sum_{j=1}^{\min(n,m)} q_j \, c_{n-j}$$

or, since $q_0 = 1$,

$$p_n = \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j}.$$

We also give an example for non-diagonal Padé approximant, with $n = 5$ and $m = 3$. We have:

$$(q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + c_7 z^7 + c_8 z^8) - \\ -(p_5 z^5 + p_4 z^4 + p_3 z^3 + p_2 z^2 + p_1 z + p_0) = \mathcal{O}\left(z^9\right).$$

Again, we compare the degrees:

$$\begin{aligned}
z^0 &: q_0 c_0 - p_0 = 0 \\
z^1 &: q_1 c_0 + q_0 c_1 - p_1 = 0 \\
z^2 &: q_2 c_0 + q_1 c_1 + q_0 c_2 - p_2 = 0 \\
z^3 &: q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 - p_3 = 0 \\
z^4 &: q_3 c_1 + q_2 c_2 + q_1 c_3 + q_0 c_4 - p_4 = 0 \\
z^5 &: q_3 c_2 + q_2 c_3 + q_1 c_4 + q_0 c_5 - p_5 = 0 \\
z^6 &: q_3 c_3 + q_2 c_4 + q_1 c_5 + q_0 c_6 = 0 \\
z^7 &: q_3 c_4 + q_2 c_5 + q_1 c_6 + q_0 c_7 = 0 \\
z^8 &: q_3 c_5 + q_2 c_6 + q_1 c_7 + q_0 c_8 = 0
\end{aligned}$$

As before, the first five equations give the coefficients of $P_{n,m}$. Again, we set $q_0 = 1$, and we write the last three equations as a matrix:

$$\begin{pmatrix} c_3 & c_4 & c_5 \\ c_4 & c_5 & c_6 \\ c_5 & c_6 & c_7 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} -c_6 \\ -c_7 \\ -c_8 \end{pmatrix}.$$

We notice that the index of the coefficients of the Taylor expansion has changed (compared to the diagonal case). Solving this matrix equation yields the coefficients of $Q_{n,m}$. The formula for the coefficients of $P_{n,m}$ is the same as before:

$$p_n = \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j}.$$

Notice that the first four coefficients, namely p_0, p_1, p_2, p_3, are computed by the same way as in the diagonal case. For the last two we apply the formula $\min(n, m)$ – the sum is from index 0 to 3.

This algorithm can be applied for all $n >= m$, with $m >= 20$. Computation of lower degrees requires that $n \gg m$, say $n = 90$, $m = 10$; otherwise the results might be incorrect.

Conclusion: The input for the Padé approximant is a function, which has a Taylor expansion around the origin (is holomorphic at the origin); the output are two polynomials of degrees n and m, respectively.

2 Hermite-Padé approximant of $[1, f, g]$

Let the functions $f(z)$ and $g(z)$ be given and their Taylor expansions be:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \quad g(z) = \sum_{j=0}^{\infty} d_j z^j.$$

Let the polynomials $P \in \Pi_n$, $Q \in \Pi_m$, $R \in \Pi_l$ have no common divisor and satisfy the condition:

$$P(z) + Q(z)f(z) + R(z)g(z) = \mathcal{O}\left(z^{n+m+l+2}\right).$$

Define $[a_1, a_2, a_3]$ as a "collection" of three objects. We set $[P, Q, R]$ for the Hermite-Padé polynomials for the collection of the three functions $[1, f, g]$. Therefore we have that:

$$\begin{aligned} &p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0 + \\ &\quad + (q_m z^m + q_{m-1} z^{m-1} + \ldots + q_0)(c_0 + c_1 z + \ldots + c_{n+m+l+1} z^{n+m+l+1}) + \\ &\quad + (r_l z^l + r_{l-1} z^{l-1} + \ldots + r_0)(d_0 + d_1 z + \ldots + d_{n+m+l+1} z^{n+m+l+1}) = \mathcal{O}\left(z^{n+m+l+2}\right). \end{aligned}$$

For example, let $n = m = l = 3$. Then:

$$\begin{aligned} &p_3 z^3 + p_2 z^2 + p_1 z + p_0 + \\ &\quad + (q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + \ldots + c_{10} z^{10}) + \\ &\quad + (r_3 z^3 + r_2 z^2 + r_1 z + r_0)(d_0 + d_1 z + d_2 z^2 + \ldots + d_{10} z^{10}) = \mathcal{O}\left(z^{11}\right). \end{aligned}$$

Comparing each degree, we have:

$$\begin{aligned} z^0 &: p_0 + q_0 c_0 + r_0 d_0 = 0 \\ z^1 &: p_1 + q_1 c_0 + q_0 c_1 + r_1 d_0 + r_0 d_1 = 0 \\ z^2 &: p_2 + q_2 c_0 + q_1 c_1 + q_0 c_2 + r_2 d_0 + r_1 d_1 + r_0 d_2 = 0 \\ z^3 &: p_3 + q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 + r_3 d_0 + r_2 d_1 + r_1 d_2 + r_0 d_3 = 0 \\ z^4 &: q_3 c_1 + q_2 c_2 + q_1 c_3 + q_0 c_4 + r_3 d_1 + r_2 d_2 + r_1 d_3 + r_0 d_4 = 0 \\ z^5 &: q_3 c_2 + q_2 c_3 + q_1 c_4 + q_0 c_5 + r_3 d_2 + r_2 d_3 + r_1 d_4 + r_0 d_5 = 0 \\ z^6 &: q_3 c_3 + q_2 c_4 + q_1 c_5 + q_0 c_6 + r_3 d_3 + r_2 d_4 + r_1 d_5 + r_0 d_6 = 0 \\ z^7 &: q_3 c_4 + q_2 c_5 + q_1 c_6 + q_0 c_7 + r_3 d_4 + r_2 d_5 + r_1 d_6 + r_0 d_7 = 0 \\ z^8 &: q_3 c_5 + q_2 c_6 + q_1 c_7 + q_0 c_8 + r_3 d_5 + r_2 d_6 + r_1 d_7 + r_0 d_8 = 0 \\ z^9 &: q_3 c_6 + q_2 c_7 + q_1 c_8 + q_0 c_9 + r_3 d_6 + r_2 d_7 + r_1 d_8 + r_0 d_9 = 0 \\ z^{10} &: q_3 c_7 + q_2 c_8 + q_1 c_9 + q_0 c_{10} + r_3 d_7 + r_2 d_8 + r_1 d_9 + r_0 d_{10} = 0 \end{aligned}$$

The first four equations are enough to compute the coefficients of P. The last seven equations have eight variables. We set $r_0 = 1$, and we write the seven equations as a matrix:

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 \\ c_2 & c_3 & c_4 & c_5 & d_2 & d_3 & d_4 \\ c_3 & c_4 & c_5 & c_6 & d_3 & d_4 & d_5 \\ c_4 & c_5 & c_6 & c_7 & d_4 & d_5 & d_6 \\ c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 \\ c_6 & c_7 & c_8 & c_9 & d_6 & d_7 & d_8 \\ c_7 & c_8 & c_9 & c_{10} & d_7 & d_8 & d_9 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \\ q_0 \\ r_3 \\ r_2 \\ r_1 \end{pmatrix} = \begin{pmatrix} -d_4 \\ -d_5 \\ -d_6 \\ -d_7 \\ -d_8 \\ -d_9 \\ -d_{10} \end{pmatrix}.$$

We solve this matrix equation and we obtain the coefficients of Q and of R. We compute the coefficients of P with this formula (we have $r_0 = 1$):

$$p_n = -\left(\sum_{j=0}^{\min(n,l)} r_j \, d_{n-j} + \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j} \right).$$

We look at another example, let $n = 7$, $m = 5$, $l = 3$. We have:

$$\begin{aligned} & p_7 z^7 + p_6 z^6 + p_5 z^5 + p_4 z^4 + p_3 z^3 + p_2 z^2 + p_1 z + p_0 + \\ & \quad + (q_5 z^5 + q_4 z^4 + q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + \ldots + c_{16} z^{16}) + \\ & \quad + (r_3 z^3 + r_2 z^2 + r_1 z + r_0)(d_0 + d_1 z + d_2 z^2 + \ldots + d_{16} z^{16}) = \mathcal{O}\left(z^{17}\right). \end{aligned}$$

Comparing each degree, we have:

$$\begin{aligned}
z^0 &: p_0 + q_0 c_0 + r_0 d_0 = 0 \\
z^1 &: p_1 + q_1 c_0 + q_0 c_1 + r_1 d_0 + r_0 d_1 = 0 \\
z^2 &: p_2 + q_2 c_0 + q_1 c_1 + q_0 c_2 + r_2 d_0 + r_1 d_1 + r_0 d_2 = 0 \\
z^3 &: p_3 + q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 + r_3 d_0 + r_2 d_1 + r_1 d_2 + r_0 d_3 = 0 \\
z^4 &: p_4 + q_4 c_0 + q_3 c_1 + q_2 c_2 + q_1 c_3 + q_0 c_4 + r_3 d_1 + r_2 d_2 + r_1 d_3 + r_0 d_4 = 0 \\
z^5 &: p_5 + q_5 c_0 + q_4 c_1 + q_3 c_2 + q_2 c_3 + q_1 c_4 + q_0 c_5 + r_3 d_2 + r_2 d_3 + r_1 d_4 + r_0 d_5 = 0 \\
z^6 &: p_6 + q_5 c_1 + q_4 c_2 + q_3 c_3 + q_2 c_4 + q_1 c_5 + q_0 c_6 + r_3 d_3 + r_2 d_4 + r_1 d_5 + r_0 d_6 = 0 \\
z^7 &: p_7 + q_5 c_2 + q_4 c_3 + q_3 c_4 + q_2 c_5 + q_1 c_6 + q_0 c_7 + r_3 d_4 + r_2 d_5 + r_1 d_6 + r_0 d_7 = 0 \\
z^8 &: q_5 c_3 + q_4 c_4 + q_3 c_5 + q_2 c_6 + q_1 c_7 + q_0 c_8 + r_3 d_5 + r_2 d_6 + r_1 d_7 + r_0 d_8 = 0 \\
z^9 &: q_5 c_4 + q_4 c_5 + q_3 c_6 + q_2 c_7 + q_1 c_8 + q_0 c_9 + r_3 d_6 + r_2 d_7 + r_1 d_8 + r_0 d_9 = 0 \\
z^{10} &: q_5 c_5 + q_4 c_6 + q_3 c_7 + q_2 c_8 + q_1 c_9 + q_0 c_{10} + r_3 d_7 + r_2 d_8 + r_1 d_9 + r_0 d_{10} = 0 \\
z^{11} &: q_5 c_6 + q_4 c_7 + q_3 c_8 + q_2 c_9 + q_1 c_{10} + q_0 c_{11} + r_3 d_8 + r_2 d_9 + r_1 d_{10} + r_0 d_{11} = 0 \\
z^{12} &: q_5 c_7 + q_4 c_8 + q_3 c_9 + q_2 c_{10} + q_1 c_{11} + q_0 c_{12} + r_3 d_9 + r_2 d_{10} + r_1 d_{11} + r_0 d_{12} = 0 \\
z^{13} &: q_5 c_8 + q_4 c_9 + q_3 c_{10} + q_2 c_{11} + q_1 c_{12} + q_0 c_{13} + r_3 d_{10} + r_2 d_{11} + r_1 d_{12} + r_0 d_{13} = 0 \\
z^{14} &: q_5 c_9 + q_4 c_{10} + q_3 c_{11} + q_2 c_{12} + q_1 c_{13} + q_0 c_{14} + r_3 d_{11} + r_2 d_{12} + r_1 d_{13} + r_0 d_{14} = 0 \\
z^{15} &: q_5 c_{10} + q_4 c_{11} + q_3 c_{12} + q_2 c_{13} + q_1 c_{14} + q_0 c_{15} + r_3 d_{12} + r_2 d_{13} + r_1 d_{14} + r_0 d_{15} = 0 \\
z^{16} &: q_5 c_{11} + q_4 c_{12} + q_3 c_{13} + q_2 c_{14} + q_1 c_{15} + q_0 c_{16} + r_3 d_{13} + r_2 d_{14} + r_1 d_{15} + r_0 d_{16} = 0
\end{aligned}$$

The first eight equations give the coefficients of P. The last nine equations have ten variables. Again, we set $r_0 = 1$, and we write the last nine equations as a matrix:

$$\begin{pmatrix}
c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 \\
c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & d_6 & d_7 & d_8 \\
c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & d_7 & d_8 & d_9 \\
c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & d_8 & d_9 & d_{10} \\
c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & d_9 & d_{10} & d_{11} \\
c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & d_{10} & d_{11} & d_{12} \\
c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & d_{11} & d_{12} & d_{13} \\
c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & d_{12} & d_{13} & d_{14} \\
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & d_{13} & d_{14} & d_{15}
\end{pmatrix}
\begin{pmatrix} q_5 \\ q_4 \\ q_3 \\ q_2 \\ q_1 \\ q_0 \\ r_3 \\ r_2 \\ r_1 \end{pmatrix}
=
\begin{pmatrix} -d_8 \\ -d_9 \\ -d_{10} \\ -d_{11} \\ -d_{12} \\ -d_{13} \\ -d_{14} \\ -d_{15} \\ -d_{16} \end{pmatrix}.$$

Solving this matrix equation yields the coefficients of Q and of R. The formula for the coefficients of P is the same as before:

$$p_n = -\left(\sum_{j=0}^{\min(n,l)} r_j \, d_{n-j} + \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j} \right).$$

This algorithm can be applied for all $n >= m$, $n >= l$, with $m, l >= 20$. Computation of lower degrees requires that $n \gg m$ and $n \gg l$, say $n = 90$, $m = 10$, $l = 15$; otherwise the results might be incorrect.

Conclusion: The input for the Hermite-Padé approximant are two functions, which have a Taylor expansion around the origin (are holomorphic at the origin); the output are three polynomials of degrees n, m and l, respectively.

3 Hermite-Padé approximant of $[1, f, g, h]$

Let the functions $f(z)$, $g(z)$ and $h(z)$ be given and their Taylor expansions be:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \quad g(z) = \sum_{j=0}^{\infty} d_j z^j, \quad h(z) = \sum_{j=0}^{\infty} e_j z^j.$$

Let the polynomials $P \in \Pi_n$, $Q \in \Pi_m$, $R \in \Pi_l$, $S \in \Pi_k$ have no common divisor and satisfy the condition:

$$P(z) + Q(z)f(z) + R(z)g(z) + S(z)h(z) = \mathcal{O}\left(z^{n+m+l+k+3}\right).$$

Define $[a_1, a_2, a_3, a_4]$ as a "collection" of four objects. We set $[P, Q, R, S]$ for the Hermite-Padé polynomials for the collection of the four functions $[1, f, g, h]$. Therefore we have that:

$$\begin{aligned}
&p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0 + \\
&\quad + (q_m z^m + q_{m-1} z^{m-1} + \ldots + q_0)(c_0 + c_1 z + \ldots + c_{n+m+l+k+2} z^{n+m+l+k+2}) + \\
&\quad + (r_l z^l + r_{l-1} z^{l-1} + \ldots + r_0)(d_0 + d_1 z + \ldots + d_{n+m+l+k+2} z^{n+m+l+k+2}) + \\
&\quad + (s_k z^k + s_{k-1} z^{k-1} + \ldots + s_0)(e_0 + e_1 z + \ldots + e_{n+m+l+k+2} z^{n+m+l+k+2}) = \\
&\quad = \mathcal{O}\left(z^{n+m+l+k+3}\right).
\end{aligned}$$

This is analogous to the previous case – one more polynomial is added to the matrix. This time we set $s_0 = 1$. The coefficients of P are computed by:

$$p_n = -\left(\sum_{j=0}^{\min(n,k)} s_j \, e_{n-j} + \sum_{j=0}^{\min(n,l)} r_j \, d_{n-j} + \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j} \right).$$

Conclusion: The input for the Hermite-Padé approximant are four functions, which have a Taylor expansion around the origin (are holomorphic at the origin); the output are four polynomials of degrees n, m, l and k, respectively.

4 Two-point Padé approximant of $f(z)$ and $g(z)$

The full definition of two-point Padé approximant can be found in [2, §1.1], [3, §1.2.3], [4, §1], [5, §2]. We focus only on the case that can be implemented in software and computed.

Let the function $f(z)$ be given, with a Taylor expansion around zero:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j,$$

and the function $g(z)$ with a Taylor expansion around infinity:

$$g(z) = \sum_{j=0}^{\infty} \frac{d_j}{z^j}.$$

Let the two polynomials $P_n, Q_n \in \Pi_n$ have no common divisor, $Q_n \neq 0$, and satisfy the conditions:

$$\begin{aligned} Q_n(z)f(z) - P_n(z) &= \mathcal{O}(z^{n+1}), \\ Q_n^{\infty}(z)g(z) - P_n^{\infty}(z) &= \mathcal{O}\left(\frac{1}{z^n}\right), \end{aligned}$$

where $P_n^{\infty}(z) = P_n(1/z)$, $Q_n^{\infty}(z) = Q_n(1/z)$. We refer to this case as "case $/n+1, n/$". Alternatively,

$$\begin{aligned} Q_n(z)f(z) - P_n(z) &= \mathcal{O}(z^{n}), \\ Q_n^{\infty}(z)g(z) - P_n^{\infty}(z) &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \end{aligned}$$

and refer to it as "case $/n, n+1/$". We set:

$$\pi_{[n,n]}(z) := \frac{P_n(z)}{Q_n(z)} = \frac{p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0}{q_n z^n + q_{n-1} z^{n-1} + \ldots + q_0}$$

as the two-point Padé approximant at a finite point (including zero), and set:

$$\pi^{\infty}_{[n,n]}(z) := \frac{P_n^{\infty}(z)}{Q_n^{\infty}(z)} = \frac{p_n + \frac{p_{n-1}}{z} + \ldots + \frac{p_0}{z^n}}{q_n + \frac{q_{n-1}}{z} + \ldots + \frac{q_0}{z^n}}$$

as the two-point Padé approximant at the infinity point.

Further on, we abbreviate two-point Padé approximant as TPPA.

The first two subsections describe the algorithm for computing the TPPA, the other ones are the two examples from which the algorithm was derived.

4.1 Algorithm for case $/n+1, n/$

This algorithm works for all $n > 1$. We describe the algorithm for $n = 3$. Let:

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \mathcal{O}(z^4)$$

and

$$g(z) = d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right).$$

Then for the TPPA we have:

$$\pi_{[3,3]}(z) = \frac{P_3(z)}{Q_3(z)} = \frac{p_3 z^3 + p_2 z^2 + p_1 z + p_0}{q_3 z^3 + q_2 z^2 + q_1 z + q_0}$$

and

$$\pi^{\infty}_{[3,3]}(z) = \frac{P^{\infty}_3(z)}{Q^{\infty}_3(z)} = \frac{p_3 + \frac{p_2}{z} + \frac{p_1}{z^2} + \frac{p_0}{z^3}}{q_3 + \frac{q_2}{z} + \frac{q_1}{z^2} + \frac{q_0}{z^3}}.$$

From the definition of TPPA we have:

$$\begin{aligned} Q_3(z)f(z) - P_3(z) &= \mathcal{O}(z^4), \\ Q^{\infty}_3(z)g(z) - P^{\infty}_3(z) &= \mathcal{O}\left(\frac{1}{z^3}\right). \end{aligned}$$

We use the first equation,

$$(q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + c_3 z^3) - (p_3 z^3 + p_2 z^2 + p_1 z + p_0) = \mathcal{O}(z^4),$$

which gives the system

$$\begin{aligned} z^0 &: \quad q_0 c_0 - p_0 = 0, \\ z^1 &: \quad q_1 c_0 + q_0 c_1 - p_1 = 0, \\ z^2 &: \quad q_2 c_0 + q_1 c_1 + q_0 c_2 - p_2 = 0, \\ z^3 &: \quad q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 - p_3 = 0. \end{aligned} \tag{4.1}$$

The second equation,

$$\left(q_3 + \frac{q_2}{z} + \frac{q_1}{z^2} + \frac{q_0}{z^3}\right)\left(d_0 + \frac{d_1}{z} + \frac{d_2}{z^2}\right) - \left(p_3 + \frac{p_2}{z} + \frac{p_1}{z^2} + \frac{p_0}{z^3}\right) = \mathcal{O}\left(\frac{1}{z^3}\right),$$

gives the system

$$\begin{aligned} z^0 &: \quad q_3 d_0 - p_3 = 0, \\ z^{-1} &: \quad q_3 d_1 + q_2 d_0 - p_2 = 0, \\ z^{-2} &: \quad q_3 d_2 + q_2 d_1 + q_1 d_0 - p_1 = 0. \end{aligned} \tag{4.2}$$

We have 7 equations and 8 variables. We set $q_0 = 1$, then $p_0 = c_0$. Substitute $p_{1,2,3}$ from the second system into the first and write the remaining equations:

$$\left|\begin{aligned} & q_3(c_0 - d_0) + q_2 c_1 + q_1 c_2 + c_3 = 0, \\ & -q_3 d_1 + q_2(c_0 - d_0) + q_1 c_1 + c_2 = 0, \\ & -q_3 d_2 - q_2 d_1 + q_1(c_0 - d_0) + c_1 = 0. \end{aligned}\right.$$

The matrix representation is:

$$\begin{pmatrix} c_0 - d_0 & c_1 & c_2 \\ -d_1 & c_0 - d_0 & c_1 \\ -d_2 & -d_1 & c_0 - d_0 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} -c_3 \\ -c_2 \\ -c_1 \end{pmatrix}.$$

Solving the matrix equation returns the coefficients of Q_3. The formula for the coefficients of P_n is:

$$p_n = \sum_{j=0}^{n} q_j \, c_{n-j}.$$

This formula corresponds to the first system of equations (4.1) above and is equivalent to the formula from classical Padé approximant. We can also use the second system of equations (4.2) for the coefficients of P_n, then:

$$p_n = \sum_{j=0}^{N-n} q_{n+j} \, d_j,$$

where $N = n$ is the maximal order and does not change when computing. For example, p_1 from (4.2) is computed by:

$$p_1 = \sum_{j=0}^{3-1} q_{1+j} \, d_j = q_1 d_0 + q_2 d_1 + q_3 d_2.$$

Conclusion: The input for the two-point Padé approximant are two functions, which have a Taylor expansion, one of them around the origin, and the other around the infinity point; the output are two polynomials of degrees n and m, respectively.

4.2 Algorithm for case $/n, n+1/$

Now we swap the maximal order of the two functions. Let $n = 3$,

$$f(z) = c_0 + c_1 z + c_2 z^2 + \mathcal{O}(z^3)$$

and

$$g(z) = d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \frac{d_3}{z^3} + \mathcal{O}\left(\frac{1}{z^4}\right).$$

From the definition of TPPA we have:

$$\begin{aligned} Q_3(z)f(z) - P_3(z) &= \mathcal{O}(z^3) \\ Q_3^{\infty}(z)g(z) - P_3^{\infty}(z) &= \mathcal{O}\left(\frac{1}{z^4}\right). \end{aligned}$$

Again, we take the first equation,

$$(q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2) - (p_3 z^3 + p_2 z^2 + p_1 z + p_0) = \mathcal{O}(z^3),$$

which gives the system

$$\begin{aligned} z^0 &: \quad q_0 c_0 - p_0 = 0, \\ z^1 &: \quad q_1 c_0 + q_0 c_1 - p_1 = 0, \\ z^2 &: \quad q_2 c_0 + q_1 c_1 + q_0 c_2 - p_2 = 0. \end{aligned} \tag{4.3}$$

The second equation,

$$\left(q_3 + \frac{q_2}{z} + \frac{q_1}{z^2} + \frac{q_0}{z^3}\right)\left(d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \frac{d_3}{z^3}\right) - \left(p_3 + \frac{p_2}{z} + \frac{p_1}{z^2} + \frac{p_0}{z^3}\right) = \mathcal{O}\left(\frac{1}{z^3}\right),$$

gives the system

$$\begin{aligned} z^0 &: \quad q_3 d_0 - p_3 = 0, \\ z^{-1} &: \quad q_3 d_1 + q_2 d_0 - p_2 = 0, \\ z^{-2} &: \quad q_3 d_2 + q_2 d_1 + q_1 d_0 - p_1 = 0, \\ z^{-3} &: \quad q_3 d_3 + q_2 d_2 + q_1 d_1 + q_0 d_0 - p_0 = 0. \end{aligned} \tag{4.4}$$

Again, we have 7 equations and 8 variables. This time we set $q_3 = 1$, then $p_3 = d_0$. (We can set again $q_0 = 1$, but then the calculations are equivalent to the previous algorithm.) We substitute $p_{0,1,2}$ from the second system into the first (this way we get the same matrix as before):

$$\left|\begin{aligned} &q_2(c_0 - d_0) + q_1 c_1 + q_0 c_2 - d_1 = 0, \\ &-q_2 d_1 + q_1(c_0 - d_0) + q_0 c_1 - d_2 = 0, \\ &-q_2 d_2 - q_1 d_1 + q_0(c_0 - d_0) - d_1 = 0. \end{aligned}\right.$$

The matrix representation is:

$$\begin{pmatrix} c_0 - d_0 & c_1 & c_2 \\ -d_1 & c_0 - d_0 & c_1 \\ -d_2 & -d_1 & c_0 - d_0 \end{pmatrix} \begin{pmatrix} q_2 \\ q_1 \\ q_0 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

The matrix on the left is the same as in the previous section, but the two vectors are different. Therefore the results from the calculations differ slightly.

Solving the matrix equation returns the coefficients of Q_3. The formula for the coefficients of P_n is:

$$p_n = \sum_{j=0}^{n} q_j \, c_{n-j}.$$

This formula corresponds to the first system of equations (4.3) above and is equivalent to the formula from classical Padé approximant. We can also use the second system of equations (4.4) for the coefficients of P_n, then:

$$p_n = \sum_{j=0}^{N-n} q_{n+j} \, d_j,$$

where $N = n$ is the maximal order and does not change when computing. For example, p_2 from (4.4) is computed by:

$$p_2 = \sum_{j=0}^{3-2} q_{2+j} \, d_j = q_2 d_0 + q_3 d_1.$$

Conclusion: The input for the two-point Padé approximant are two functions, which have a Taylor expansion, one of them around the origin, and the other around the infinity point; the output are two polynomials of degrees n and m, respectively.

4.3 Example 1

This example is from [2, §1.1]. Let:

$$f(z) = \left(\frac{1+\frac{1}{2}z}{1+2z}\right)^{1/2},$$

which has expansion around zero of

$$f(z) = 1 - \frac{3}{4}z + \frac{39}{32}z^2 - \frac{267}{128}z^3 + \frac{7563}{2048}z^4 + \mathcal{O}(z^5).$$

We map $z = 1/t$ and obtain:

$$f(t) = \left(\frac{2t+1}{2t+4}\right)^{1/2},$$

which has expansion around zero (the expansion around infinity of $f(z)$) of

$$f(t) = \frac{1}{2} + \frac{3}{8}t - \frac{21}{64}t^2 + \frac{87}{256}t^3 - \frac{1677}{4096}t^4 + \mathcal{O}(z^5).$$

Let $n = 1$ and take the expansion at zero with maximal order $n + 1$

$$f(z) = 1 - \frac{3}{4}z + \mathcal{O}(z^2)$$

and the expansion at infinity with maximal order n

$$f(z) = \frac{1}{2} + \mathcal{O}\left(\frac{1}{z}\right).$$

Therefore we have that

$$\pi_{[1,1]}(z) = \frac{P_1(z)}{Q_1(z)} = \frac{p_1 z + p_0}{q_1 z + q_0}, \quad \pi^{\infty}_{[1,1]}(z) = \frac{P^{\infty}_1(z)}{Q^{\infty}_1(z)} = \frac{p_1 + \frac{p_0}{z}}{q_1 + \frac{q_0}{z}}.$$

Then the expansion at zero,

$$(q_1 z + q_0)\left(1 - \frac{3}{4}z\right) - (p_1 z + p_0) = \mathcal{O}(z^2),$$

gives the system

$$\begin{aligned} z^0: &\quad q_0 - p_0 = 0, \\ z^1: &\quad q_1 - \frac{3}{4}q_0 - p_1 = 0. \end{aligned}$$

The expansion at infinity,

$$\left(q_1 + \frac{q_0}{z}\right)\frac{1}{2} - \left(p_1 + \frac{p_0}{z}\right) = \mathcal{O}\left(\frac{1}{z}\right),$$

gives the equation

$$z^0: \quad \frac{1}{2}q_1 - p_1 = 0.$$

We have 3 equations and 4 variables. Set $q_0 = 1$, then $p_0 = 1$, $q_1 = 3/2$ and $p_1 = 3/4$. The TPPA is

$$\pi_{[1,1]}(z) = \frac{1 + 3z/4}{1 + 3z/2}.$$

4.4 Example 2

This example is from [6]. Let $n = 4$ and denote with t_0 an expansion (of some function) at zero with maximal order $n + 1$

$$t_0 = 1 - \frac{2}{3}z + \frac{4}{15}z^2 - \frac{8}{105}z^3 + \frac{16}{945}z^4 + \mathcal{O}(z^5),$$

and an expansion at infinity with maximal order n

$$t_\infty = \frac{1}{2z} + \frac{1}{4z^2} + \frac{3}{8z^3} + \mathcal{O}\left(\frac{1}{z^4}\right).$$

We have that

$$\pi_{[4,4]}(z) = \frac{P_4(z)}{Q_4(z)} = \frac{p_4z^4 + p_3z^3 + p_2z^2 + p_1z + p_0}{q_4z^4 + q_3z^3 + q_2z^2 + q_1z + q_0},$$
$$\pi^\infty_{[4,4]}(z) = \frac{P^\infty_4(z)}{Q^\infty_4(z)} = \frac{p_4 + \frac{p_3}{z} + \frac{p_2}{z^2} + \frac{p_1}{z^3} + \frac{p_0}{z^4}}{q_4 + \frac{q_3}{z} + \frac{q_2}{z^2} + \frac{q_1}{z^3} + \frac{q_0}{z^4}}.$$

From the definition, the expansion at zero $Q_4(z)t_0 - P_4(z) = \mathcal{O}(z^5)$,

$$(q_4z^4 + q_3z^3 + q_2z^2 + q_1z + q_0)\left(1 - \frac{2}{3}z + \frac{4}{15}z^2 - \frac{8}{105}z^3 + \frac{16}{945}z^4\right) - \\ -(p_4z^4 + p_3z^3 + p_2z^2 + p_1z + p_0) = \mathcal{O}(z^5),$$

gives the system

$$\begin{aligned}
z^0: &\quad q_0 - p_0 = 0,\\
z^1: &\quad q_1 - \frac{2}{3}q_0 - p_1 = 0,\\
z^2: &\quad q_2 - \frac{2}{3}q_1 + \frac{4}{15}q_0 - p_2 = 0,\\
z^3: &\quad q_3 - \frac{2}{3}q_2 + \frac{4}{15}q_1 - \frac{8}{105}q_0 - p_3 = 0,\\
z^4: &\quad q_4 - \frac{2}{3}q_3 + \frac{4}{15}q_2 - \frac{8}{105}q_1 + \frac{16}{945}q_0 - p_4 = 0.
\end{aligned}$$

Also from the definition, the expansion at infinity $Q^\infty_4(z)t_\infty - P^\infty_4(z) = \mathcal{O}(1/z^4)$,

$$\left(q_4 + \frac{q_3}{z} + \frac{q_2}{z^2} + \frac{q_1}{z^3} + \frac{q_0}{z^4}\right)\left(\frac{1}{2z} + \frac{1}{4z^2} + \frac{3}{8z^3}\right) - \\ - \left(p_4 + \frac{p_3}{z} + \frac{p_2}{z^2} + \frac{p_1}{z^3} + \frac{p_0}{z^4}\right) = \mathcal{O}\left(\frac{1}{z^4}\right),$$

gives the system (the expansion does not have degree zero):

$$\begin{aligned}
z^{-1}: &\quad \frac{1}{2}q_4 - p_3 = 0,\\
z^{-2}: &\quad \frac{1}{4}q_4 + \frac{1}{2}q_3 - p_2 = 0,\\
z^{-3}: &\quad \frac{3}{8}q_4 + \frac{1}{4}q_3 + \frac{1}{2}q_2 - p_1 = 0.
\end{aligned}$$

We have 8 equations and 10 variables. Set $q_0 = 1$, then $p_0 = 1$. The remaining equations are:

$$\begin{aligned}
z^1: &\quad q_1 - \frac{2}{3} - p_1 = 0,\\
z^2: &\quad q_2 - \frac{2}{3}q_1 + \frac{4}{15} - p_2 = 0,\\
z^3: &\quad q_3 - \frac{2}{3}q_2 + \frac{4}{15}q_1 - \frac{8}{105} - p_3 = 0,\\
z^4: &\quad q_4 - \frac{2}{3}q_3 + \frac{4}{15}q_2 - \frac{8}{105}q_1 + \frac{16}{945} - p_4 = 0,\\
z^{-1}: &\quad \frac{1}{2}q_4 - p_3 = 0,\\
z^{-2}: &\quad \frac{1}{4}q_4 + \frac{1}{2}q_3 - p_2 = 0,\\
z^{-3}: &\quad \frac{3}{8}q_4 + \frac{1}{4}q_3 + \frac{1}{2}q_2 - p_1 = 0.
\end{aligned}$$

We have 7 equations and 8 variables. Set $p_4 = 0$ (or any other variable, we do not have enough equations). Write again the equation with degree z^4, then write degree z^3 minus z^{-1}; z^2 minus z^{-2}; finally z minus z^{-3}:

$$\left|\begin{aligned}
&q_4 - \frac{2}{3}q_3 + \frac{4}{15}q_2 - \frac{8}{105}q_1 + \frac{16}{945} = 0,\\
&-\frac{1}{2}q_4 + q_3 - \frac{2}{3}q_2 + \frac{4}{15}q_1 - \frac{8}{105} = 0,\\
&-\frac{1}{4}q_4 - \frac{1}{2}q_3 + q_2 - \frac{2}{3}q_1 + \frac{4}{15} = 0,\\
&-\frac{3}{8}q_4 - \frac{1}{4}q_3 - \frac{1}{2}q_2 + q_1 - \frac{2}{3} = 0.
\end{aligned}\right.$$

The solution of the system is: $q_1 = \frac{8}{9}$, $q_2 = \frac{8}{21}$, $q_3 = \frac{32}{315}$, $q_4 = \frac{16}{945}$. We find the remaining coefficients: $p_1 = \frac{2}{9}$, $p_2 = \frac{52}{945}$, $p_3 = \frac{8}{945}$. Finally, the TPPA is:

$$\pi_{[4,4]}(z) = \frac{1 + \frac{2}{9}z + \frac{52}{945}z^2 + \frac{8}{945}z^3}{1 + \frac{8}{9}z + \frac{8}{21}z^2 + \frac{32}{315}z^3 + \frac{16}{945}z^4}.$$

5 Multipoint Padé approximant of $f(z)$

Let the function $f(z)$ be holomorphic on a compact set $E \subset \mathbb{C}$, denote by $f(z) \in \mathcal{H}(E)$. Let the polynomials $P_{n,m} \in \Pi_n$, $Q_{n,m} \in \Pi_m$ have no common divisor, $Q_{n,m} \neq 0$, and satisfy the condition:

$$\frac{Q_{n,m}(z)f(z) - P_{n,m}(z)}{\omega_{n+m+1}(z)} \in \mathcal{H}(E),$$

where z_i are interpolation points,

$$\omega_{n+m+1}(z) = \prod_{i=0}^{n+m}(z - z_i).$$

We set:

$$R_{n,m}(z) = \frac{P_{n,m}(z)}{Q_{n,m}(z)} = \frac{p_n z^n + p_{n-1}z^{n-1} + \ldots + p_0}{q_m z^m + q_{m-1}z^{m-1} + \ldots + q_0}$$

for the multipoint Padé approximant, we have that $R_{n,m}$ interpolates to the function $f(z)$ in the points z_i [2, §1.1],

$$R_{n,m}(z_i) = f(z_i), \quad i = 0, 1, 2, \ldots, n+m.$$

This equation can be written as:

$$p_n z_i^n + p_{n-1}z_i^{n-1} + \ldots + p_0 = f(z_i)(q_m z_i^m + q_{m-1}z_i^{m-1} + \ldots + q_0).$$

For example, let $n = m = 3$. Then:

$$\omega_7(z) = \prod_{i=0}^{6}(z - z_i)$$

and

$$p_3 z_i^3 + p_2 z_i^2 + p_1 z_i + p_0 = f(z_i)(q_3 z_i^3 + q_2 z_i^2 + q_1 z_i + q_0).$$

Denote by $c_i = f(z_i)$, $i = 0, 1, 2, \ldots, 6$. We rearrange the variables:

$$p_0 + p_1 z_i + p_2 z_i^2 + p_3 z_i^3 - q_1 c_i z_i - q_2 c_i z_i^2 - q_3 c_i z_i^3 = q_0 c_i.$$

We have 7 equations and 8 variables. We set $q_0 = 1$ and write the system of equations as a matrix:

$$\begin{pmatrix} 1 & z_0 & z_0^2 & z_0^3 & -c_0 z_0 & -c_0 z_0^2 & -c_0 z_0^3 \\ 1 & z_1 & z_1^2 & z_1^3 & -c_1 z_1 & -c_1 z_1^2 & -c_1 z_1^3 \\ 1 & z_2 & z_2^2 & z_2^3 & -c_2 z_2 & -c_2 z_2^2 & -c_2 z_2^3 \\ 1 & z_3 & z_3^2 & z_3^3 & -c_3 z_3 & -c_3 z_3^2 & -c_3 z_3^3 \\ 1 & z_4 & z_4^2 & z_4^3 & -c_4 z_4 & -c_4 z_4^2 & -c_4 z_4^3 \\ 1 & z_5 & z_5^2 & z_5^3 & -c_5 z_5 & -c_5 z_5^2 & -c_5 z_5^3 \\ 1 & z_6 & z_6^2 & z_6^3 & -c_6 z_6 & -c_6 z_6^2 & -c_6 z_6^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix}.$$

Solving this matrix equation returns the coefficients of $P_{n,m}$ and of $Q_{n,m}$.

We also look at one non-diagonal case, let $n = 5$, $m = 3$. Then

$$\omega_9(z) = \prod_{i=0}^{8}(z - z_i)$$

and

$$p_5 z_i^5 + p_4 z_i^4 + p_3 z_i^3 + p_2 z_i^2 + p_1 z_i + p_0 = f(z_i)(q_3 z_i^3 + q_2 z_i^2 + q_1 z_i + q_0).$$

Again, denote by $c_i = f(z_i)$, $i = 0, 1, 2, \ldots, 8$ and rearrange the variables:

$$p_0 + p_1 z_i + p_2 z_i^2 + p_3 z_i^3 + p_4 z_i^4 + p_5 z_i^5 - q_1 c_i z_i - q_2 c_i z_i^2 - q_3 c_i z_i^3 = q_0 c_i.$$

We have 9 equations and 10 variables. Again, we set $q_0 = 1$ and we write the system of equations as a matrix:

$$\begin{pmatrix} 1 & z_0 & z_0^2 & z_0^3 & z_0^4 & z_0^5 & -c_0 z_0 & -c_0 z_0^2 & -c_0 z_0^3 \\ 1 & z_1 & z_1^2 & z_1^3 & z_1^4 & z_1^5 & -c_1 z_1 & -c_1 z_1^2 & -c_1 z_1^3 \\ 1 & z_2 & z_2^2 & z_2^3 & z_2^4 & z_2^5 & -c_2 z_2 & -c_2 z_2^2 & -c_2 z_2^3 \\ 1 & z_3 & z_3^2 & z_3^3 & z_3^4 & z_3^5 & -c_3 z_3 & -c_3 z_3^2 & -c_3 z_3^3 \\ 1 & z_4 & z_4^2 & z_4^3 & z_4^4 & z_4^5 & -c_4 z_4 & -c_4 z_4^2 & -c_4 z_4^3 \\ 1 & z_5 & z_5^2 & z_5^3 & z_5^4 & z_5^5 & -c_5 z_5 & -c_5 z_5^2 & -c_5 z_5^3 \\ 1 & z_6 & z_6^2 & z_6^3 & z_6^4 & z_6^5 & -c_6 z_6 & -c_6 z_6^2 & -c_6 z_6^3 \\ 1 & z_7 & z_7^2 & z_7^3 & z_7^4 & z_7^5 & -c_7 z_7 & -c_7 z_7^2 & -c_7 z_7^3 \\ 1 & z_8 & z_8^2 & z_8^3 & z_8^4 & z_8^5 & -c_8 z_8 & -c_8 z_8^2 & -c_8 z_8^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \end{pmatrix}.$$

As before, solving this matrix equation returns the coefficients of $P_{n,m}$ and of $Q_{n,m}$.

Conclusion: The input for the multipoint Padé approximant are interpolation points, their count is $n + m + 1$, and a function, which does not turn to zero at the interpolation points (is holomorphic at the interpolation points); the output are two polynomials of degrees n and m, respectively.

The requirement is that the function does not turn zero at the interpolation points. Therefore, the above conditions can be relaxed considerably: let the function $f(z)$ be m-meromorphic (has no more than m poles) on a compact set $E \subset \mathbb{C}$, denote by $f(z) \in \mathcal{M}_m(E)$; then the condition is:

$$\frac{Q_{n,m}(z) f(z) - P_{n,m}(z)}{\omega_{n+m+1}(z)} \in \mathcal{M}_m(E).$$

For example, let $E = C_0(1)$ be the unit circle, take interpolation points on ∂E with $\omega_{n+m+1} = z^{n+m+1} - 1$, and let the function be:

$$f(z) = \frac{z}{\sqrt{2 - z^2}} + \frac{1}{(1 - 2.45z)^{10}}.$$

The function has a pole of order 10 at $(0.408, 0)$ and $f(z) \in \mathcal{M}_{10}(E)$. At most 10 poles of the multipoint Padé approximant converge to that pole, the rest are in $\mathbb{C} \setminus C_0(\sqrt{2})$, therefore $R_{n,m} \in \mathcal{M}_{10}(E)$, where $R_{n,m} = P_{n,m}/Q_{n,m}$.

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