Viskovatov algorithm for Hermite–Padé approximants

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Padé approximant of $f(z)$

Let the function $f(z)$ be given and its Taylor expansion be:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j.$$

Let the two polynomials $P_{n,m} \in \Pi_n$, $Q_{n,m} \in \Pi_m$ have no common divisor, $Q_{n,m} \neq 0$, and satisfy the condition:

$$Q_{n,m}(z)f(z) - P_{n,m}(z) = \mathcal{O}\left(z^{n+m+1}\right).$$

We set

$$\pi_{n,m}(z) = \frac{P_{n,m}(z)}{Q_{n,m}(z)} = \frac{p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0}{q_m z^m + q_{m-1} z^{m-1} + \ldots + q_0}$$

for the Padé approximant of $f(z)$. Therefore we have that:

$$(q_m z^m + q_{m-1} z^{m-1} + \ldots + q_0)(c_0 + c_1 z + \ldots + c_{n+m} z^{n+m}) - \\ -(p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0) = \mathcal{O}\left(z^{n+m+1}\right).$$

For example, let $n = m = 3$. Then:

$$(q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6) - \\ -(p_3 z^3 + p_2 z^2 + p_1 z + p_0) = \mathcal{O}\left(z^7\right).$$

Comparing each degree, we have:

$$\begin{aligned}
z^0 &: q_0 c_0 - p_0 = 0 \\
z^1 &: q_1 c_0 + q_0 c_1 - p_1 = 0 \\
z^2 &: q_2 c_0 + q_1 c_1 + q_0 c_2 - p_2 = 0 \\
z^3 &: q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 - p_3 = 0 \\
z^4 &: q_3 c_1 + q_2 c_2 + q_1 c_3 + q_0 c_4 = 0 \\
z^5 &: q_3 c_2 + q_2 c_3 + q_1 c_4 + q_0 c_5 = 0 \\
z^6 &: q_3 c_3 + q_2 c_4 + q_1 c_5 + q_0 c_6 = 0
\end{aligned}$$

The first four equations are enough to compute the coefficients of $P_{n,m}$. The last three equations have four variables. We set $q_0 = 1$, and we write the three equations as a matrix:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} -c_4 \\ -c_5 \\ -c_6 \end{pmatrix}.$$

We solve this matrix equations and we have the coefficients of $Q_{n,m}$. Now we compute the coefficients of $P_{n,m}$ with this formula:

$$p_n = c_n + \sum_{j=1}^{\min(n,m)} q_j \, c_{n-j}$$

or, since $q_0 = 1$,

$$p_n = \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j}.$$

Hermite-Padé approximant of $[1, f, g]$

Let the functions $f(z)$ and $g(z)$ be given and their Taylor expansions be:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j, \quad g(z) = \sum_{j=0}^{\infty} d_j z^j.$$

Let the polynomials $P \in \Pi_n$, $Q \in \Pi_m$, $R \in \Pi_l$ have no common divisor and satisfy the condition:

$$P(z) + Q(z)f(z) + R(z)g(z) = \mathcal{O}\left(z^{n+m+l+2}\right).$$

Define $[a_1, a_2, a_3]$ as a "collection" of three objects. We set $[P, Q, R]$ for the Hermite-Padé polynomials for the collection of the three functions $[1, f, g]$. Therefore we have that:

$$p_n z^n + p_{n-1} z^{n-1} + \ldots + p_0 +$$
$$+ (q_m z^m + q_{m-1} z^{m-1} + \ldots + q_0)(c_0 + c_1 z + \ldots + c_{n+m+l+1} z^{n+m+l+1}) +$$
$$+ (r_l z^l + r_{l-1} z^{l-1} + \ldots + r_0)(d_0 + d_1 z + \ldots + d_{n+m+l+1} z^{n+m+l+1}) = \mathcal{O}\left(z^{n+m+l+2}\right)$$

For example, let $n = m = l = 3$. Then:

$$p_3 z^3 + p_2 z^2 + p_1 z + p_0 + \\ + (q_3 z^3 + q_2 z^2 + q_1 z + q_0)(c_0 + c_1 z + c_2 z^2 + \ldots + c_{10} z^{10}) + \\ + (r_3 z^3 + r_2 z^2 + r_1 z + r_0)(d_0 + d_1 z + d_2 z^2 + \ldots + d_{10} z^{10}) = \mathcal{O}\left(z^{11}\right).$$

Comparing each degree, we have:

$$z^0 : p_0 + q_0 c_0 + r_0 d_0 = 0$$

$$z^1 : p_1 + q_1 c_0 + q_0 c_1 + r_1 d_0 + r_0 d_1 = 0$$

$$z^2 : p_2 + q_2 c_0 + q_1 c_1 + q_0 c_2 + r_2 d_0 + r_1 d_1 + r_0 d_2 = 0$$

$$z^3 : p_3 + q_3 c_0 + q_2 c_1 + q_1 c_2 + q_0 c_3 + r_3 d_0 + r_2 d_1 + r_1 d_2 + r_0 d_3 = 0$$

$$z^4 : q_3 c_1 + q_2 c_2 + q_1 c_3 + q_0 c_4 + r_3 d_1 + r_2 d_2 + r_1 d_3 + r_0 d_4 = 0$$

$$z^5 : q_3 c_2 + q_2 c_3 + q_1 c_4 + q_0 c_5 + r_3 d_2 + r_2 d_3 + r_1 d_4 + r_0 d_5 = 0$$

$$z^6 : q_3 c_3 + q_2 c_4 + q_1 c_5 + q_0 c_6 + r_3 d_3 + r_2 d_4 + r_1 d_5 + r_0 d_6 = 0$$

$$z^7 : q_3 c_4 + q_2 c_5 + q_1 c_6 + q_0 c_7 + r_3 d_4 + r_2 d_5 + r_1 d_6 + r_0 d_7 = 0$$

$$z^8 : q_3 c_5 + q_2 c_6 + q_1 c_7 + q_0 c_8 + r_3 d_5 + r_2 d_6 + r_1 d_7 + r_0 d_8 = 0$$

$$z^9 : q_3 c_6 + q_2 c_7 + q_1 c_8 + q_0 c_9 + r_3 d_6 + r_2 d_7 + r_1 d_8 + r_0 d_9 = 0$$

$$z^{10} : q_3 c_7 + q_2 c_8 + q_1 c_9 + q_0 c_{10} + r_3 d_7 + r_2 d_8 + r_1 d_9 + r_0 d_{10} = 0$$

The first four equations are enough to compute the coefficients of P. The last seven equations have eight variables. We set $r_0 = 1$, and we write the seven equations as a matrix:

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 \\ c_2 & c_3 & c_4 & c_5 & d_2 & d_3 & d_4 \\ c_3 & c_4 & c_5 & c_6 & d_3 & d_4 & d_5 \\ c_4 & c_5 & c_6 & c_7 & d_4 & d_5 & d_6 \\ c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 \\ c_6 & c_7 & c_8 & c_9 & d_6 & d_7 & d_8 \\ c_7 & c_8 & c_9 & c_{10} & d_7 & d_8 & d_9 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \\ q_0 \\ r_3 \\ r_2 \\ r_1 \end{pmatrix} = \begin{pmatrix} -d_4 \\ -d_5 \\ -d_6 \\ -d_7 \\ -d_8 \\ -d_9 \\ -d_{10} \end{pmatrix}.$$

We solve this matrix equation and we obtain the coefficients of Q and of R. We compute the coefficients of P with this formula (we have $r_0 = 1$):

$$p_n = -\left(\sum_{j=0}^{\min(n,l)} r_j \, d_{n-j} + \sum_{j=0}^{\min(n,m)} q_j \, c_{n-j} \right).$$

Continued fraction

Regular C-fraction:

$$c = 1 + \cfrac{z}{2 + \cfrac{z}{3 + \cfrac{z}{4 + \cfrac{z}{5 + \cfrac{z}{6 + \cfrac{z}{7}}}}}}.$$

Result is:

$$c = \frac{16z^3 + 600z^2 + 4320z + 5040}{z^3 + 120z^2 - 1800z + 5040}.$$

Viskovatov's algorithm for $\exp(z)$

$$f_1 = \exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5)$$
$$f_0 = 1 + O(z^1)$$

Zero coefficient:

$$c_1 = 1 \quad (\text{from } f_1)$$
$$c_0 = 1 \quad (\text{from } f_0)$$

First coefficient:

$$f_1^{[1]} = f_0 = 1 + O(z^1)$$
$$f_0^{[1]} = \frac{1}{z}\left(f_1 - \frac{c_1}{c_0} f_0\right) = \frac{1}{z}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5) - \frac{1}{1}\left(1 + O(z^1)\right)\right) =$$
$$= 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4)$$
$$c_1^{[1]} = 1 \qquad (c_1^{[k]} = \text{zero coefficient from } f_1^{[k]},\ k \geq 1)$$
$$c_0^{[1]} = 1 \qquad (c_0^{[k]} = \text{zero coefficient from } f_0^{[k]},\ k \geq 1)$$

Second coefficient:

$$
\begin{aligned}
f_1^{[2]} &= f_0^{[1]} = 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4) \\
f_0^{[2]} &= \frac{1}{z}\left(f_1^{[1]} - \frac{c_1^{[1]}}{c_0^{[1]}} f_0^{[1]}\right) = \\
&= \frac{1}{z}\left(1 + O(z^1) - \frac{1}{1}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4)\right)\right) = \\
&= \frac{1}{z}\left(1 - 1 - \frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} + O(z^4)\right) = \\
&= -\frac{1}{2} - \frac{z}{6} - \frac{z^2}{24} + O(z^3) \\
c_1^{[2]} &= 1 \\
c_0^{[2]} &= -\frac{1}{2}
\end{aligned}
$$

Third coefficient:

$$f_1^{[3]} = f_0^{[2]} = -\frac{1}{2} - \frac{z}{6} - \frac{z^2}{24} + O(z^3)$$

$$f_0^{[3]} = \frac{1}{z}\left(f_1^{[2]} - \frac{c_1^{[2]}}{c_0^{[2]}} f_0^{[2]}\right) =$$

$$= \frac{1}{z}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4) - \frac{1}{-1/2}\left(-\frac{1}{2} - \frac{z}{6} - \frac{z^2}{24} + O(z^3)\right)\right) =$$

$$= \frac{1}{z}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4) - 1 - \frac{z}{3} - \frac{z^2}{12} + O(z^3)\right)$$

$$= \frac{1}{6} + \frac{z}{12} + O(z^2)$$

$$c_1^{[3]} = -\frac{1}{2}$$

$$c_0^{[3]} = \frac{1}{6}$$

The coefficients for the continued fraction are $c_1^{[k]}/c_0^{[k]}$, $k = 1, 2, 3, \ldots$

$$\text{coeff}_0 = \frac{c_1}{c_0} = 1$$

$$\text{coeff}_1 = \frac{c_1^{[1]}}{c_0^{[1]}} = 1$$

$$\text{coeff}_2 = \frac{c_1^{[2]}}{c_0^{[2]}} = \frac{1}{-1/2} = -2$$

$$\text{coeff}_3 = \frac{c_1^{[3]}}{c_0^{[3]}} = \frac{-1/2}{1/6} = -3$$

By computer program, the first eight are:

$$\overrightarrow{c_1} = \left[1, 1, -\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{60}, -\frac{1}{120}, \frac{1}{840}\right]$$

$$\overrightarrow{c_0} = \left[1, -\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{60}, -\frac{1}{120}, \frac{1}{840}, \frac{1}{1680}\right]$$

$$\overrightarrow{c_1/c_0} = [1, -2, -3, 2, 5, -2, -7, 2]$$

Continued fraction is:

$$\frac{f_1}{f_0} = 1 + \frac{z}{1} + \frac{z}{-2} + \frac{z}{-3} + \frac{z}{2} + \frac{z}{5} + \frac{z}{-2} + \frac{z}{-7} + \frac{z}{2}.$$

Alternative writing:

$$\frac{f_1}{f_0} = 1 + \cfrac{z}{1 + \cfrac{z}{-2 + \cfrac{z}{-3 + \cfrac{z}{2 + \cfrac{z}{5 + \cfrac{z}{-2 + \cfrac{z}{-7 + \frac{z}{2}}}}}}}}.$$

Padé approximant is:

$$[4/4] = \frac{z^4 + 20z^3 + 180z^2 + 840z + 1680}{z^4 - 20z^3 + 180z^2 - 840z + 1680}.$$

Viskovatov's algorithm for $[1, \exp(z), \exp(z)^{1/3}]$

Viskovatov's algorithm for $[1, \exp(z), \exp(z)^{1/3}]$:

$$f_2 = \exp(z)^{1/3} = 1 + \frac{z}{3} + \frac{z^2}{18} + \frac{z^3}{162} + \frac{z^4}{1944} + O(z^5)$$
$$f_1 = \exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5)$$
$$f_0 = 1 + O(z^1)$$

Zero coefficient:

$$c_2 = 1 \quad (\text{from } f_2)$$
$$c_1 = 1 \quad (\text{from } f_1)$$
$$c_0 = 1 \quad (\text{from } f_0)$$

First coefficient:

$$f_2^{[1]} = f_0 = 1 + O(z^1)$$

$$f_1^{[1]} = \frac{1}{z}\left(f_2 - \frac{c_2}{c_1} f_1\right) =$$

$$= \frac{1}{z}\left(1 + \frac{z}{3} + \frac{z^2}{18} + \frac{z^3}{162} + O(z^4) - \frac{1}{1}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5)\right)\right) =$$

$$= -\frac{2}{3} - \frac{4z}{9} - \frac{13z^2}{81} - \frac{10z^3}{243} - \frac{121z^4}{14580} + O(z^5)$$

$$f_0^{[1]} = \frac{1}{z}\left(f_1 - \frac{c_1}{c_0} f_0\right) = \frac{1}{z}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5) - \frac{1}{1}\left(1 + O(z^1)\right)\right) =$$

$$= 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4)$$

$$c_2^{[1]} = 1 \qquad (c_2^{[k]} = \text{zero coefficient from } f_2^{[k]},\ k \geq 1)$$

$$c_1^{[1]} = -\frac{2}{3} \qquad (c_1^{[k]} = \text{zero coefficient from } f_1^{[k]},\ k \geq 1)$$

$$c_0^{[1]} = 1 \qquad (c_0^{[k]} = \text{zero coefficient from } f_0^{[k]},\ k \geq 1)$$

Second coefficient:

$$f_2^{[2]} = f_0^{[1]} = 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4)$$

$$f_1^{[2]} = \frac{1}{z}\left(f_2^{[1]} - \frac{c_2^{[1]}}{c_1^{[1]}} f_1^{[1]}\right) =$$

$$= \frac{1}{z}\left(1 + O(z^1) + \frac{3}{2}\left(-\frac{2}{3} - \frac{4z}{9} - \frac{13z^2}{81} - \frac{10z^3}{243} - \frac{121z^4}{14580} + O(z^5)\right)\right) =$$

$$= -\frac{2}{3} - \frac{13z}{54} - \frac{5z^2}{81} - \frac{121z^3}{9720} - \frac{91z^4}{43740} + O(z^5)$$

$$f_0^{[2]} = \frac{1}{z}\left(f_1^{[1]} - \frac{c_1^{[1]}}{c_0^{[1]}} f_0^{[1]}\right) =$$

$$= \frac{1}{z}\left(-\frac{2}{3} - \frac{4z}{9} - \frac{13z^2}{81} - \frac{10z^3}{243} + O(z^4) + \frac{2}{3}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4)\right)\right) =$$

$$= -\frac{1}{9} - \frac{4z}{81} - \frac{13z^2}{972} - \frac{2z^3}{729} + O(z^4)$$

$$c_2^{[2]} = 1 \qquad c_1^{[2]} = -\frac{2}{3} \qquad c_0^{[2]} = -\frac{1}{9}$$

Third coefficient:

$$f_2^{[3]} = f_0^{[2]} = -\frac{1}{9} - \frac{4z}{81} - \frac{13z^2}{972} - \frac{2z^3}{729} + O(z^4)$$

$$f_1^{[3]} = \frac{1}{z}\left(f_2^{[2]} - \frac{c_2^{[2]}}{c_1^{[2]}} f_1^{[2]}\right) =$$

$$= \frac{1}{z}\left(1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + O(z^4) + \frac{3}{2}\left(-\frac{2}{3} - \frac{13z}{54} - \frac{5z^2}{81} - \frac{121z^3}{9720} + O(z^4)\right)\right)$$

$$= \frac{5}{36} + \frac{2z}{27} + \frac{149z^2}{6480} + \frac{19z^3}{3645} + \frac{2309z^4}{2449440} + O(z^5)$$

$$f_0^{[3]} = \frac{1}{z}\left(f_1^{[2]} - \frac{c_1^{[2]}}{c_0^{[2]}} f_0^{[2]}\right) =$$

$$= \frac{1}{z}\left(-\frac{2}{3} - \frac{13z}{54} - \frac{5z^2}{81} + O(z^3) - 6\left(-\frac{1}{9} - \frac{4z}{81} - \frac{13z^2}{972} - \frac{2z^3}{729} + O(z^4)\right)\right) =$$

$$= \frac{1}{18} + \frac{z}{54} - \frac{13z^2}{3240} - \frac{z^3}{1458} + O(z^4)$$

$$c_2^{[3]} = -\frac{1}{9} \qquad c_1^{[3]} = \frac{5}{36} \qquad c_0^{[3]} = \frac{1}{18}$$

By computer program, the first eight are:

$$\overrightarrow{c_2} = \left[1, 1, -\frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \frac{1}{2430}, \frac{1}{3240}, \frac{1}{5670}, -\frac{1}{1837080}\right]$$

$$\overrightarrow{c_1} = \left[-\frac{2}{3}, -\frac{2}{3}, \frac{5}{36}, \frac{4}{405}, -\frac{7}{720}, \frac{17}{11340}, -\frac{29}{1301265}, -\frac{31}{986580}\right]$$

$$\overrightarrow{c_0} = \left[1, -\frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \frac{1}{2430}, \frac{1}{3240}, \frac{1}{5670}, -\frac{1}{1837080}\right]$$

We apply the following formulas:

$$c_q^{[k]} = -\frac{c_2^{[k]}}{c_1^{[k]}}, \qquad c_r^{[k]} = -\frac{c_1^{[k]}}{c_0^{[k]}}$$

$$\overrightarrow{c_q} = \left[\frac{3}{2}, \frac{3}{2}, \frac{4}{5}, -\frac{45}{8}, \frac{20}{7}, -\frac{14}{51}, \frac{3213}{232}, \frac{174}{31}\right]$$

$$\overrightarrow{c_r} = \left[\frac{2}{3}, -6, -\frac{5}{2}, -\frac{16}{45}, \frac{189}{8}, -\frac{34}{7}, \frac{58}{459}, -\frac{1674}{29}\right]$$

Then we define even more variables!

Zero coefficient:

$$c_q^0 = -\frac{c_2}{c_1}$$
$$c_r^0 = -\frac{c_1}{c_0}$$

First coefficient:

$$Q_1^{[1]} = c_q^{[1]}$$
$$Q_2^{[1]} = c_q^{[1]} c_q^0$$
$$Q_3^{[1]} = z$$
$$R_1^{[1]} = 1$$
$$R_2^{[1]} = c_r^{[1]} + c_q^0$$
$$R_3^{[1]} = c_r^{[1]} c_r^0$$

Second coefficient:

$$
\begin{aligned}
Q_1^{[2]} &= c_q^{[2]} Q_1^{[1]} + 0z \\
Q_2^{[2]} &= c_q^{[2]} Q_2^{[1]} + 1z \\
Q_3^{[2]} &= c_q^{[2]} Q_3^{[1]} + c_r^0 z \\
R_1^{[2]} &= c_r^{[2]} R_1^{[1]} + Q_1^{[1]} \\
R_2^{[2]} &= c_r^{[2]} R_2^{[1]} + Q_2^{[1]} \\
R_3^{[2]} &= c_r^{[2]} R_3^{[1]} + Q_3^{[1]}
\end{aligned}
$$

Finally, this code is repeated for $k \geq 3$:

$$
\begin{aligned}
Q_1^{[k]} &= c_q^{[k]} Q_1^{[k-1]} + z R_1^{[k-2]} \\
Q_2^{[k]} &= c_q^{[k]} Q_2^{[k-1]} + z R_2^{[k-2]} \\
Q_3^{[k]} &= c_q^{[k]} Q_3^{[k-1]} + z R_3^{[k-2]} \\
R_1^{[k]} &= c_r^{[k]} R_1^{[k-1]} + Q_1^{[k-1]} \\
R_2^{[k]} &= c_r^{[k]} R_2^{[k-1]} + Q_2^{[k-1]} \\
R_3^{[k]} &= c_r^{[k]} R_3^{[k-1]} + Q_3^{[k-1]}
\end{aligned}
$$

This is the final stage.

The result is in $3n+1$ iteration, where $n=4$ in this particular case. This results in 3 polynomials, each of degree $n=4$.

$$R_3^{[13]} = -\frac{32}{243}z^4 - \frac{2560}{243}z^3 - \frac{3200}{9}z^2 - \frac{481280}{81}z - \frac{3331840}{81}$$
$$R_2^{[13]} = -\frac{1}{243}z^4 + \frac{50}{243}z^3 - \frac{115}{27}z^2 + \frac{3475}{81}z - \frac{14270}{81}$$
$$R_1^{[13]} = z^4 - 30z^3 + 675z^2 - 7695z + 41310$$

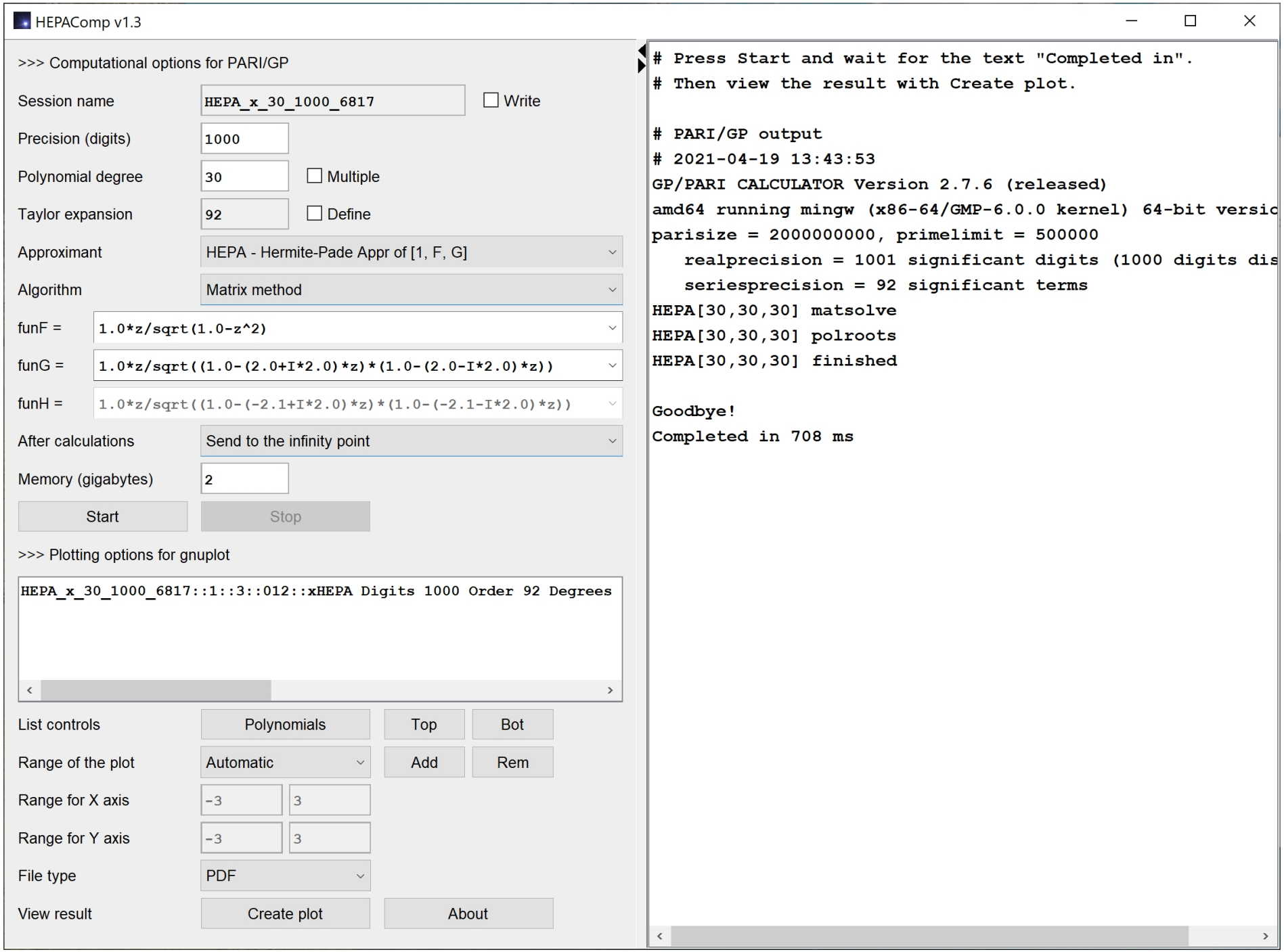
This is the end, this is known as Hermite–Padé approximant of type I for the collection of functions $[1, f, g] = [1, \exp(z), \exp(z)^{1/3}]$.

There exists type II which is computed by type I so we are required to do these calculations.

What was the point of all this? We already have a working matrix method. Why do we need a new Viskovatov algorithm?

The answer is speed of the calculations. The matrix method becomes quite slow for degrees $n \geq 400$. Even that is already pushing the memory limits and it takes about 24–48 hours to compute.

The new Viskovatov algorithm is 10 times faster while using about the same amount of computer memory.


HEPAComp v1.3
>>> Computational options for PARI/GP
Session name
HEPA_x_30_1000_6817
Write
Precision (digits)
1000
Polynomial degree
30
Multiple
Taylor expansion
92
Define
Approximant
HEPA - Hermite-Pade Appr of [1, F, G]
Algorithm
Matrix method
funF =
1.0*z/sqrt(1.0-z^2)
funG =
1.0*z/sqrt((1.0-(2.0+I*2.0)*z)*(1.0-(2.0-I*2.0)*z))
funH =
1.0*z/sqrt((1.0-(-2.1+I*2.0)*z)*(1.0-(-2.1-I*2.0)*z))
After calculations
Send to the infinity point
Memory (gigabytes)
2
Start
Stop
>>> Plotting options for gnuplot
HEPA_x_30_1000_6817::1::3::012::xHEPA Digits 1000 Order 92 Degrees
List controls
Polynomials
Top
Bot
Range of the plot
Automatic
Add
Rem
Range for X axis
-3
3
Range for Y axis
-3
3
File type
PDF
View result
Create plot
About
Press Start and wait for the text "Completed in".
Then view the result with Create plot.
PARI/GP output
2021-04-19 13:43:53
GP/PARI CALCULATOR Version 2.7.6 (released)
amd64 running mingw (x86-64/GMP-6.0.0 kernel) 64-bit versic
parisize = 2000000000, primelimit = 500000
realprecision = 1001 significant digits (1000 digits dis
seriesprecision = 92 significant terms
HEPA[30,30,30] matsolve
HEPA[30,30,30] polroots
HEPA[30,30,30] finished
Goodbye!
Completed in 708 ms


George A. Baker, Jr., Peter Graves-Morris, *Padé approximants*, Second Edition, Cambridge University Press, New York, 1996.


N. R. Ikonomov, S. P. Suetin, Viskovatov algorithm for Hermite-Padé polynomials, arXiv:2007.03370 [math.CV].

Software is available here:
`http://justmathbg.info/`

Thank you for the attention!

Padé approximant

Let $n = m = 3$. Let $Q_3(z)f(z) - P_3(z) = \mathcal{O}(z^7)$, where $f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_6 z^6$.

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} -c_4 \\ -c_5 \\ -c_6 \end{pmatrix}.$$

We set $q_0 = 1$ and we obtain all coefficients of $Q_3(z)$, the denominator. The numerator is computed by the coefficients of $Q_3(z)$.

Another method to obtain the coefficients:

$$Q_3^*(z) = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \\ c_3 & c_4 & c_5 & c_6 \\ z^3 & z^2 & z & 1 \end{vmatrix}.$$

We obtain $Q_3^*(z)$, and after normalization it is equal to $Q_3(z)$:

$$Q_3^*(z)/q_0^* = Q_3(z).$$

We can generalize both of the above approaches by [1, §8.5 Hermite-Padé polynomials, Equation 5.4]:

$$Q_3^*(z) = \begin{vmatrix} 1 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c_1 & c_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c_2 & c_1 & c_0 & 0 \\ 0 & 0 & 0 & 1 & c_3 & c_2 & c_1 & c_0 \\ 0 & 0 & 0 & 0 & c_4 & c_3 & c_2 & c_1 \\ 0 & 0 & 0 & 0 & c_5 & c_4 & c_3 & c_2 \\ 0 & 0 & 0 & 0 & c_6 & c_5 & c_4 & c_3 \\ 0 & 0 & 0 & 0 & 1 & z & z^2 & z^3 \end{vmatrix}.$$

The matrix equation that we solve by software is much smaller than the full determinant, while the result is the same.

Computing the numerator:

$$P_3^*(z) = \begin{vmatrix} 1 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c_1 & c_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c_2 & c_1 & c_0 & 0 \\ 0 & 0 & 0 & 1 & c_3 & c_2 & c_1 & c_0 \\ 0 & 0 & 0 & 0 & c_4 & c_3 & c_2 & c_1 \\ 0 & 0 & 0 & 0 & c_5 & c_4 & c_3 & c_2 \\ 0 & 0 & 0 & 0 & c_6 & c_5 & c_4 & c_3 \\ 1 & z & z^2 & z^3 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Normalization: $P_3^*(z)/p_0^* = P_3(z)/p_0$
($p_0 = c_0$, but might not be equal to 1).

Hermite-Padé approximant

Let $n = m = l = 3$. Let $P_3(z) + Q_3(z)f(z) + R_3(z)g(z) = \mathcal{O}(z^{11})$, where:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_{10} z^{10}, \quad g(z) = d_0 + d_1 z + d_2 z^2 + \ldots + d_{10} z^{10}.$$

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 \\ c_2 & c_3 & c_4 & c_5 & d_2 & d_3 & d_4 \\ c_3 & c_4 & c_5 & c_6 & d_3 & d_4 & d_5 \\ c_4 & c_5 & c_6 & c_7 & d_4 & d_5 & d_6 \\ c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 \\ c_6 & c_7 & c_8 & c_9 & d_6 & d_7 & d_8 \\ c_7 & c_8 & c_9 & c_{10} & d_7 & d_8 & d_9 \end{pmatrix} \begin{pmatrix} q_3 \\ q_2 \\ q_1 \\ q_0 \\ r_3 \\ r_2 \\ r_1 \end{pmatrix} = \begin{pmatrix} -d_4 \\ -d_5 \\ -d_6 \\ -d_7 \\ -d_8 \\ -d_9 \\ -d_{10} \end{pmatrix}.$$

We set $r_0 = 1$ and we obtain all coefficients of $Q_3(z)$ and $R_3(z)$. The coefficients of $P_3(z)$ are computed by the former ones.

Another method to obtain the coefficients:

$$R_3^*(z) = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 & d_4 \\ c_2 & c_3 & c_4 & c_5 & d_2 & d_3 & d_4 & d_5 \\ c_3 & c_4 & c_5 & c_6 & d_3 & d_4 & d_5 & d_6 \\ c_4 & c_5 & c_6 & c_7 & d_4 & d_5 & d_6 & d_7 \\ c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 & d_8 \\ c_6 & c_7 & c_8 & c_9 & d_6 & d_7 & d_8 & d_9 \\ c_7 & c_8 & c_9 & c_{10} & d_7 & d_8 & d_9 & d_{10} \\ 0 & 0 & 0 & 0 & z^3 & z^2 & z & 1 \end{vmatrix}$$

$$Q_3^*(z) = \begin{vmatrix} c_1 & c_2 & c_3 & c_4 & d_1 & d_2 & d_3 & d_4 \\ c_2 & c_3 & c_4 & c_5 & d_2 & d_3 & d_4 & d_5 \\ c_3 & c_4 & c_5 & c_6 & d_3 & d_4 & d_5 & d_6 \\ c_4 & c_5 & c_6 & c_7 & d_4 & d_5 & d_6 & d_7 \\ c_5 & c_6 & c_7 & c_8 & d_5 & d_6 & d_7 & d_8 \\ c_6 & c_7 & c_8 & c_9 & d_6 & d_7 & d_8 & d_9 \\ c_7 & c_8 & c_9 & c_{10} & d_7 & d_8 & d_9 & d_{10} \\ z^3 & z^2 & z & 1 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

After normalization we obtain the following formulas (we have $r_0 = 1$ and $q_0 \neq 1$):

$$R_3^*(z)/r_0^* = R_3(z), \quad Q_3^*(z)/q_0^* = Q_3(z)/q_0.$$

As above, we can generalize both approaches by [1, §8.5 Hermite-Padé polynomials, Equation 5.4]:

$$R_3^*(z) = \left|\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & d_0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & c_1 & c_0 & 0 & 0 & d_1 & d_0 & 0 & 0 \\
0 & 0 & 1 & 0 & c_2 & c_1 & c_0 & 0 & d_2 & d_1 & d_0 & 0 \\
0 & 0 & 0 & 1 & c_3 & c_2 & c_1 & c_0 & d_3 & d_2 & d_1 & d_0 \\
0 & 0 & 0 & 0 & c_4 & c_3 & c_2 & c_1 & d_4 & d_3 & d_2 & d_1 \\
0 & 0 & 0 & 0 & c_5 & c_4 & c_3 & c_2 & d_5 & d_4 & d_3 & d_2 \\
0 & 0 & 0 & 0 & c_6 & c_5 & c_4 & c_3 & d_6 & d_5 & d_4 & d_3 \\
0 & 0 & 0 & 0 & c_7 & c_6 & c_5 & c_4 & d_7 & d_6 & d_5 & d_4 \\
0 & 0 & 0 & 0 & c_8 & c_7 & c_6 & c_5 & d_8 & d_7 & d_6 & d_5 \\
0 & 0 & 0 & 0 & c_9 & c_8 & c_7 & c_6 & d_9 & d_8 & d_7 & d_6 \\
0 & 0 & 0 & 0 & c_{10} & c_9 & c_8 & c_7 & d_{10} & d_9 & d_8 & d_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 & z^3
\end{array}\right| .$$

We have an analogous formula for $Q_3^*(z)$. The matrix equation that we solve be software is a lot smaller than the full determinant, while the result is the same.

Finally:

$$P_3^*(z) = \begin{vmatrix} 1 & 0 & 0 & 0 & c_0 & 0 & 0 & 0 & d_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c_1 & c_0 & 0 & 0 & d_1 & d_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & c_2 & c_1 & c_0 & 0 & d_2 & d_1 & d_0 & 0 \\ 0 & 0 & 0 & 1 & c_3 & c_2 & c_1 & c_0 & d_3 & d_2 & d_1 & d_0 \\ 0 & 0 & 0 & 0 & c_4 & c_3 & c_2 & c_1 & d_4 & d_3 & d_2 & d_1 \\ 0 & 0 & 0 & 0 & c_5 & c_4 & c_3 & c_2 & d_5 & d_4 & d_3 & d_2 \\ 0 & 0 & 0 & 0 & c_6 & c_5 & c_4 & c_3 & d_6 & d_5 & d_4 & d_3 \\ 0 & 0 & 0 & 0 & c_7 & c_6 & c_5 & c_4 & d_7 & d_6 & d_5 & d_4 \\ 0 & 0 & 0 & 0 & c_8 & c_7 & c_6 & c_5 & d_8 & d_7 & d_6 & d_5 \\ 0 & 0 & 0 & 0 & c_9 & c_8 & c_7 & c_6 & d_9 & d_8 & d_7 & d_6 \\ 0 & 0 & 0 & 0 & c_{10} & c_9 & c_8 & c_7 & d_{10} & d_9 & d_8 & d_7 \\ 1 & z & z^2 & z^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Normalization: $P_3^*(z)/p_0^* = P_3(z)/p_0$.