VISKOVATOV ALGORITHM FOR HERMITE–PADÉ POLYNOMIALS

N. R. IKONOMOV AND S. P. SUETIN

Abstract. We propose an algorithm for producing Hermite–Padé polynomials of type I for an arbitrary tuple of $m+1$ formal power series $[f_0,\dots,f_m]$, $m \geqslant 1$, about $z=0$ ($f_j \in \mathbb{C}[[z]]$) under the assumption that the series have a certain ('general position') nondegeneracy property. This algorithm is a straightforward extension of the classical Viskovatov algorithm for construction of Padé polynomials (for $m=1$ our algorithm coincides with the Viskovatov algorithm).

The algorithm proposed here is based on a recurrence relation and has the feature that all the Hermite–Padé polynomials corresponding to the multiindices $(k,k,k,\dots,k,k)$, $(k+1,k,k,\dots,k,k)$, $(k+1,k+1,k,\dots,k,k),\dots,$ $(k+1,k+1,k+1,\dots,k+1,k)$ are already known by the time the algorithm produces the Hermite–Padé polynomials corresponding to the multiindex $(k+1,k+1,k+1,\dots,k+1,k+1)$.

We show how the Hermite–Padé polynomials corresponding to different multiindices can be found via this algorithm by changing appropriately the initial conditions.

The algorithm can be parallelized in $m+1$ independent evaluations at each nth step.

Bibliography: [28] titles.

Keywords: formal power series, Hermite–Padé polynomials, Viskovatov algorithm.

Contents

1. The case $m=1$: a tuple of series $[f_0,f_1]$ (finding Padé polynomials) 1
2. The case $m=2$: a tuple of series $[f_0,f_1,f_2]$ 7
3. The case of an arbitrary $m\in\mathbb{N}$: a tuple of series $[f_0,\dots,f_m]$ 13
References 21

1. The case $m=1$: a tuple of series $[f_0,f_1]$ (finding Padé polynomials)

1.1. **Introduction. Classical Viskovatov algorithm.** The most well known algorithms for producing the coefficients of the expansion of a given (in general, formal) power series $f\in\mathbb{C}[[z]]$ in a continued C-fraction (this thereby also produces Padé polynomials) are the QD-algorithm (see [2, Ch. 4, § 4.3, Theorem 4.3.5]) and the Viskovatov algorithm (see [27, § 4, pp. 243–245] and [2, Ch. 4,

Date: July 7, 2020.

The research of the second author was carried out with partial financial support of the Russian Foundation for Basic Research (grant no. 18-01-00764).

§ 4.2, formula (2.17)]). Both algorithms are capable of producing Padé approximants of the form $[n/n]_f, [n+1/n]_f$, but they can be applied only under certain nondegeneracy condition of the original power series f (the series f should be in 'general position').

In the present paper, we extend the classical Viskovatov algorithm to the setting of Hermite–Padé polynomials of type I for a tuple of $m+1$ formal power series $f_0, \ldots, f_m$, where $m \geqslant 1$ and $f_j \in \mathbb{C}[[z]]$, $j = 0, \ldots, m$. For $m = 1$, for a tuple of series $[f_0, f_1]$, this algorithm (for finding Hermite–Padé polynomials) produces Padé polynomials. This is why it can be looked upon as a natural extension of the classical Viskovatov algorithm to the case of Hermite–Padé polynomials. It seems that with this algorithm one can associate the so-called vector continued fraction (for details, see [2, Ch. 8, § 8.4], [14] and the references given therein). However, this question will not be addressed in the present paper. Recurrent relations close to those discussed below and related to the extension of the Viskovatov algorithm to the case of Hermite–Padé polynomials were obtained in [19] and [14]; see also [6], [4], [18] and [5].

Note that it is the Viskovatov algorithm that ws used by A. Trias when implementing the HELM-algorithm for producing Padé polynomials (see [22], [23], and also [24]).

Let $a(z) := \sum\limits_{k=0}^{\infty} a_k z^k$, $b(z) := \sum\limits_{k=0}^{\infty} b_k z^k$ be formal power series, $b_0 \neq 0$. The Viskovatov algorithm of expansion of the ratio of the series $a(z)/b(z)$ in a continued C-fraction is based on the following identity (see [2, § 4.2, formula (2.17)]):

$$\frac{\sum\limits_{k=0}^{\infty} a_k z^k}{\sum\limits_{k=0}^{\infty} b_k z^k} = \frac{a_0}{b_0} + \frac{z}{\sum\limits_{k=0}^{\infty} b_k z^k \,\Big/ \sum\limits_{k=0}^{\infty} (a_{k+1} - a_0 b_{k+1}/b_0) z^k}. \tag{1.1}$$

An application of a similar identity to the series appearing in the denominator on the right of (1.1) constitutes the next step in the expansion of the ratio of the series $a(z)/b(z)$ in a continued C-fraction

$$v_0 + \cfrac{z}{v_1 + \cfrac{z}{v_2 + \cfrac{z}{v_3 + \ldots}}}. \tag{1.2}$$

In the Padé table for the series $f(z) := a(z)/b(z)$, the approximants of the continued C-fraction (1.2) form a staircase sequence consisting of Padé approximants of the form $[n/n]_f$ and $[n+1/n]_f$, $n = 0, 1, \ldots$.

Note that for formal Laurent series about the point at infinity ζ of the form

$$F(\zeta) := \sum_{k=0}^{\infty} \frac{c_k}{\zeta^{k+1}}, \tag{1.3}$$

expansions in a Chebyshev continued fraction are constructed using the Jacobi–Perron algorithm. There exists a multivariate analogue of the Jacobi–Perron algorithm capable of producing Hermite–Padé polynomials for a tuple of series (1.3) (see [15], [16] and the references cited there). Such algorithms also apply for producing Hermite–Padé polynomials under certain nondegeneracy conditions. One

can also mention QD-algorithms (see, first of all [25] and [26] and the bibliography given there) capable of producing Hermite–Padé polynomials for a tuple of Laurent series (1.3) under similar nondegeneracy conditions. However, these QD-algorithms for construction of Hermite–Padé polynomials pertain to formal Laurent series rather than to formal Taylor series, and, as far as we know, no link between formal Taylor series and the classical QD-algorithms has been established so far.

It is also worth noting that from the point of view of applications, the traditional interest to Hermite–Padé polynomials of type I stems mainly from the fact that they underlie the construction of Shafer quadratic approximants (also called algebraic approximants); see [19], [17], [20], [9], [10], [28], [1], [8] and the references there. Nevertheless, a new approach to the analytic continuation problem was recently proposed in [21, § 4, formulae (61)–(63)] (see also [11]); this approach is based on the use of Hermite–Padé polynomials of type I, but in it only rational functions are involved. In [11], this approach was theoretically justified for a sufficiently large class of multivalued analytic functions; for more details, see [11, formula (9), Corollary 6].

1.2. **Hermite–Padé polynomials (Padé polynomials) for a tuple of series $[f_0, f_1]$.** Here we reduce the Viskovatov algorithm to the form more convenient for further applications. We set $f_0 = f_0(z) = \sum\limits_{k=0}^{\infty} c_{0,k} z^k = c_0 + O(z)$, $f_1 = f_1(z) = \sum\limits_{k=0}^{\infty} c_{1,k} z^k = c_1 + O(z)$; here and in what follows, by $O(z)$ we denote power series starting with the first-order term in z. Hence relation (1.1) can be written as the identity

$$\frac{f_1}{f_0} = \frac{c_1}{c_0} + \frac{z}{f_0 \,/\, \left[\left(f_1 - \dfrac{c_1}{c_0} f_0\right)/z\right]} = \frac{c_1}{c_0} + \frac{z}{f_1^{[1]}/f_0^{[1]}}, \tag{1.4}$$

where we put

$$f_1^{[1]} := f_0 =: \sum_{k=0}^{\infty} c_{1,k}^{[1]} = c_1^{[1]} + O(z), \quad f_0^{[1]} := \frac{1}{z}\left(f_1 - \frac{c_1}{c_0} f_0\right) = \sum_{k=0}^{\infty} c_{0,k}^{[1]} = c_0^{[1]} + O(z).$$

Similarly to (1.4), for the new series $f_0^{[1]}$ and $f_1^{[1]}$, we set

$$\frac{f_1^{[1]}}{f_0^{[1]}} = \frac{c_1^{[1]}}{c_0^{[1]}} + \frac{z}{f_1^{[2]}/f_0^{[2]}}, \tag{1.5}$$

where

$$f_1^{[2]} := f_0^{[1]} =: \sum_{k=0}^{\infty} c_{1,k}^{[2]} z^k = c_1^{[2]} + O(z),$$

$$f_0^{[2]} := \frac{1}{z}\left(f_1^{[1]} - \frac{c_1^{[1]}}{c_0^{[1]}} f_0^{[1]}\right) = \sum_{k=0}^{\infty} c_{0,k}^{[2]} z^k = c_0^{[2]} + O(z).$$

From (1.3) and (1.5) we get the following relation, in which we put $f_0^{[0]} := f_0 =: \sum_{k=0}^{\infty} c_{0,k}^{[0]} z^k = c_0^{[0]} + O(z)$, $f_1^{[0]} := f_1 =: \sum_{k=0}^{\infty} c_{1,k}^{[0]} z^k = c_1^{[0]} + O(z)$,

$$\frac{f_1}{f_0} = \frac{f_1^{[0]}}{f_0^{[0]}} = \frac{c_1^{[0]}}{c_0^{[0]}} + \cfrac{z}{\cfrac{c_1^{[1]}}{c_0^{[1]}} + \cfrac{z}{f_1^{[2]}/f_0^{[2]}}}. \tag{1.6}$$

Applying formula of the form (1.5) to $f_1^{[2]}/f_0^{[2]}, f_1^{[3]}/f_0^{[3]}, \dots$, we get the (formal) expansion of the ratio f_1/f_0 in a continued C-fraction of the form (1.2).

1.3. **The matrix approach.** Let $f_0^{[0]} := f_0, f_1^{[0]} := f_1, f_0^{[1]}, f_1^{[1]}, \dots$ be formal series (in the above sense).

Setting

$$M_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} z & 0 \\ -c_1^{[0]}/c_0^{[0]} & 1 \end{pmatrix}, \quad \vec{f} := (f_1, f_0),$$

we have

$$M_1 \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \quad M_2 \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} zf_0 \\ f_1 - \dfrac{c_1}{c_0} f_0 \end{pmatrix} = z \begin{pmatrix} f_1^{[1]} \\ f_0^{[1]} \end{pmatrix}.$$

We define

$$M^{[0]} := M_2 M_1 = \begin{pmatrix} 0 & z \\ 1 & -c_1^{[0]}/c_0^{[0]} \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{pmatrix}, \tag{1.7}$$

where P_1, P_2, Q_1, Q_2 are polynomials of z, $P_j.Q_j \in \mathbb{C}[z]$. So, we have

$$M^{[0]} \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = z \begin{pmatrix} f_1^{[1]} \\ f_0^{[1]} \end{pmatrix} = O(z). \tag{1.8}$$

Now from (1.8) we get that $Q_1 f_1 + Q_2 f_0 = O(z)$, where $Q_1 = \text{const}$, $Q_2 = \text{const}$. Hence Q_2 and Q_1 are Hermite–Padé polynomials of type I (Padé polynomials) for the tuple of series $[f_0, f_1]$ and the multiindex $\mathbf{k} = (0, 0)$, because the order of tangency in (1.8) agrees with the multiindex $\mathbf{k}$: $|\mathbf{k}| + 1 = 0 + 1 = 1$.

As in (1.8), we have

$$M^{[1]} \begin{pmatrix} f_1^{[1]} \\ f_0^{[1]} \end{pmatrix} = z \begin{pmatrix} f_1^{[2]} \\ f_0^{[2]} \end{pmatrix}, \quad \text{where} \quad M^{[1]} = \begin{pmatrix} 0 & z \\ 1 & -c_1^{[1]}/c_0^{[1]} \end{pmatrix}.$$

Therefore,

$$M^{[1]} M^{[0]} \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = z^2 \begin{pmatrix} f_1^{[2]} \\ f_0^{[2]} \end{pmatrix} = O(z^2). \tag{1.9}$$

We set $A^{[0]} := M^{[0]}$, $A^{[1]} := M^{[1]} M^{[0]}$. Hence

$$A^{[1]} = M^{[1]} M^{[0]} = \begin{pmatrix} z & -\dfrac{c_1}{c_0} z \\ -\dfrac{c_1^{[1]}}{c_0^{[1]}} & z + \dfrac{c_1^{[1]}}{c_0^{[1]}} \dfrac{c_1}{c_0} \end{pmatrix} =: \begin{pmatrix} P_1^{[1]} & P_2^{[1]} \\ Q_1^{[1]} & Q_2^{[1]} \end{pmatrix}, \tag{1.10}$$

where $\deg Q_1^{[1]} = 0$, $\deg Q_2^{[1]} = 1$. From (1.9) it follows that $Q_2^{[1]} f_0 + Q_1^{[1]} f_1 = O(z^2)$. Therefore, $Q_2^{[1]}, Q_1^{[1]}$ is a pair of Hermite–Padé polynomials (Padé polynomials) for the tuple of series $[f_0, f_1]$ and the multiindex $\mathbf{k} = \mathbf{k}^{[1]} = (1, 0)$. Proceeding as in (1.8) and (1.9), we arrive in succession to Hermite–Padé polynomials for the tuple $[f_0, f_1]$ and the multiindices $\mathbf{k}^{[2]} = (1, 1)$, $\mathbf{k}^{[3]} = (2, 1)$, $\mathbf{k}^{[4]} = (2, 2)$, $\mathbf{k}^{[5]} = (3, 2), \ldots$.

1.4. **Hermite–Padé polynomials for a tuple of series $[f_0, f_1]$: an arbitrary iteration step $(n + 1)$.** Let $n \geqslant 1$. We set

$$f_1^{[n+1]} := f_0^{[n]} =: c_1^{[n+1]} + O(z), \quad f_0^{[n+1]} := \frac{1}{z}\left(f_1^{[n]} - \frac{c_1^{[n]}}{c_0^{[n]}} f_0^{[n]}\right) =: c_0^{[n+1]} + O(z).$$

Let

$$M^{[n]} := \begin{pmatrix} 0 & z \\ 1 & -c_1^{[n]}/c_0^{[n]} \end{pmatrix}, \quad A^{[n]} := M^{[n]} \cdots M^{[0]} = \begin{pmatrix} P_1^{[n]} & P_2^{[n]} \\ Q_1^{[n]} & Q_2^{[n]} \end{pmatrix},$$

where $P_j^{[n]}, Q_j^{[n]}$ are polynomials. Hence

$$A^{[n+1]} := M^{[n+1]} A^{[n]} = \begin{pmatrix} P_1^{[n+1]} & P_2^{[n+1]} \\ Q_1^{[n+1]} & Q_2^{[n+1]} \end{pmatrix}, \tag{1.11}$$

where $P_j^{[n+1]}, Q_j^{[n+1]} \in \mathbb{C}[z]$. By definition of the matrix $M^{[n]}$ and using (1.11), we have the recurrence relations

$$\begin{aligned} P_j^{[n+1]} &= z Q_j^{[n]}, \\ Q_j^{[n+1]} &= P_j^{[n]} - \frac{c_1^{[n+1]}}{c_0^{[n+1]}} Q_j^{[n]}, \quad j = 1, 2, \ldots, \quad n = 1, 2, \ldots. \end{aligned} \tag{1.12}$$

From (1.12) we get the *three-term* recurrence relation, which relates the polynomials $Q_j^{[n+1]}$, $j = 1, 2$, as obtained at step $(n + 1)$ of the iteration, to the polynomials obtained at two previous steps:

$$Q_j^{[n+1]}(z) = -\frac{c_1^{[n+1]}}{c_0^{[n+1]}} Q_j^{[n]}(z) + z Q_j^{[n-1]}(z), \quad j = 1, 2, \quad n = 1, 2, \ldots. \tag{1.13}$$

The initial conditions for relations (1.13) follow from (1.7) and (1.10):

$$Q_1^{[0]} = Q_1 = 1, \quad Q_2^{[0]} = Q_2 = -\frac{c_1}{c_0}, \qquad Q_1^{[1]} = -\frac{c_1^{[1]}}{c_0^{[1]}}, \quad Q_2^{[1]} = z + \frac{c_1^{[1]}}{c_0^{[1]}} \frac{c_1}{c_0}. \tag{1.14}$$

Moreover, we have

$$A^{[n]} \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = z^{n+1} \begin{pmatrix} f_1^{[n+1]} \\ f_0^{[n+1]} \end{pmatrix} = O(z^{n+1}). \tag{1.15}$$

The following result holds.

Theorem 1. *1) Let $n = 2k$, $k \geqslant 0$. Then $\deg Q_1^{[n]} = \deg Q_2^{[n]} = k$ and the polynomials $Q_2^{[n]}, Q_1^{[n]}$ are Hermite–Padé polynomials for the tuple of series $[f_0, f_1]$ and the multiindex $\mathbf{k}^{[n]} = (k, k)$ (the order of tangency is $|\mathbf{k}^{[n]}|+1 = 2k+1 = n+1$, see (1.15)).*

2) Let $n = 2k+1$, $k \geqslant 0$. Then $\deg Q_1^{[n]} = k$, $\deg Q_2^{[n]} = k+1$ and the polynomials $Q_2^{[n]}, Q_1^{[n]}$ are Hermite–Padé polynomials for the tuple of series $[f_0, f_1]$ and the multiindex $\mathbf{k}^{[n]} = (k+1, k)$ (the order of tangency is $|\mathbf{k}^{[n]}|+1 = 2k+2 = n+1$, see (1.15)).

Remark 1. Here and in what follows, in Theorems 1–3 and the corresponding algorithms, we assume that the original series $f_0, f_1, \ldots, f_m$ are in 'general position' (cf. [7]). In particular, we assume that all the polynomials appearing in the statements of Theorems 1–3 have the precise degree indicated in the theorems. In general, in the corresponding relations for the degree the nonstrict inequality should be written.

Remark 2. If we start the iteration process from the vector series ${}^{\mathrm{T}}(f_0, f_1)$ (here and in what follows ${}^{\mathrm{T}}\vec{v}$ denotes the transposition with respect to the vector row $\vec{v}$) instead of ${}^{\mathrm{T}}(f_1, f_0)$, then as a result we get the sequence of Padé approximants of the form $[n/n], [n/n+1]$, $n = 0, 1, \ldots$ (instead of $[n/n], [n+1/n]$).

Theorem 1 is in fact well known (see [2, Ch. 4, Theorem 4.2.1]; cf. also (1.13) and [2, formula (2.76)]). Nevertheless, to make our presentation complete, we give a proof of this theorem.

Proof of Theorem 1. We argue by induction on k.

1) Let $k = 0$. From (1.14), for $n = 2k = 0$ we get $\deg Q_1^{[0]} = \deg Q_2^{[0]} = 0$. Moreover, from (1.13) and (1.14) for $n = 2k+1 = 1$ we have $\deg Q_1^{[1]} = 0$, $\deg Q_2^{[1]} = 1$. For $k = 1$, using (1.13) we find that $\deg Q_1^{[2]} = \deg Q_2^{[2]} = 1$ and $\deg Q_1^{[3]} = 1$, $\deg Q_2^{[3]} = 2$.

2) Assuming now that the conclusions of Theorem 1 hold for $n = 2k-1$ and $n = 2k$, let us show that they also hold for $n = 2k+1$ and $n = 2k+2$.

For $n = 2k+1$, from the recurrence relation (1.13) we get

$$Q_1^{[2k+1]} = a^{[2k+1]} Q_1^{[2k]} + z Q_1^{[2k-1]}.$$

Therefore,

$$\deg Q_1^{[2k+1]} = \max\{\deg Q_1^{[2k]}, 1 + \deg Q_1^{[2k-1]}\} = \max\{k, 1+k-1\} = k.$$

A similar analysis shows that $Q_2^{[2k+1]} = a^{[2k+1]} Q_2^{[2k]} + z Q_2^{[2k+1]}$, and therefore, using the induction assumption,

$$\deg Q_2^{[2k+1]} = \max\{\deg Q_2^{[2k]}, 1 + \deg Q_2^{[2k-1]}\} = \max\{k, 1+k\} = k+1.$$

For $n = 2k+2$ from (1.13) we conclude that $Q_1^{[2k+2]} = a^{[2k+2]} Q_1^{[2k+1]} + z Q_1^{[2k]}$, and hence, by the induction assumption,

$$\deg Q_1^{[2k+2]} = \max\{\deg Q_1^{[2k+1]}, 1 + \deg Q_1^{[2k]}\} = \max\{k, 1+k\} = k+1.$$

In a similar manner we obtain $Q_2^{[2k+2]} = a^{[2k+2]} Q_2^{[2k+1]} + z Q_2^{[2k]}$, and therefore, by the induction assumption,

$$\deg Q_2^{[2k+2]} = \max\{\deg Q_2^{[2k+1]}, 1 + \deg Q_2^{[2k]}\} = \max\{k+1, 1+k\} = k+1.$$

Theorem 1 is proved. □

1.5. **The algorithm for $m = 1$ (a tuple of series $[f_0, f_1]$).** We are given two series $f_0 = f_0(z) = \sum\limits_{k=0}^{\infty} c_{0,k} z^k = c_0 + O(z)$ and $f_1 = f_1(z) = \sum\limits_{k=0}^{\infty} c_{1,k} z^k = c_1 + O(z)$.

1.5.1. **The initial iteration step.** If $f_0^{[0]} := f_0$, $f_1^{[0]} := f_1$, $c_0^{[0]} := c_0$, $c_1^{[0]} := c_1$, then we have $c_{0,k}^{[0]} := c_{0,k}$, $c_{1,k}^{[0]} := c_{1,k}$ for $k = 0, 1, \dots$, and $f_0^{[0]} = c_0^{[0]} + O(z)$, $f_1^{[0]} = c_1^{[0]} + O(z)$. Let $a^{[0]} := -c_1^{[0]}/c_0^{[0]}$.

1.5.2. **Step 1 of the iteration.** Setting

$$f_1^{[1]} := f_0^{[0]}, \quad f_0^{[1]} := \frac{1}{z}\left(f_1^{[0]} - \frac{c_1^{[0]}}{c_0^{[0]}} f_0^{[0]}\right), \tag{1.16}$$

we have

$$f_0^{[1]} = \sum_{k=0}^{\infty} c_{0,k}^{[1]} z^k = c_0^{[1]} + O(z), \quad f_1^{[1]} = \sum_{k=0}^{\infty} c_{1,k}^{[1]} z^k = c_1^{[1]} + O(z).$$

We set $a^{[1]} := -c_1^{[1]}/c_0^{[1]}$,

$$Q_1^{[0]} := 1, \quad Q_2^{[0]} := a^{[0]}, \quad Q_1^{[1]} := a^{[1]}, \quad Q_2^{[1]} := z + a^{[1]} a^{[0]}.$$

1.5.3. **Step $(n+1)$ of the iteration.** Let $a^{[n]} := -c_1^{[n]}/c_0^{[n]}$,

$$\begin{aligned} f_1^{[n+1]} &:= f_0^{[n]} =: \sum_{k=0}^{\infty} c_{1,k}^{[n+1]} z^k = c_1^{[n+1]} + O(z), \\ f_0^{[n+1]} &:= \frac{1}{z}\left(f_1^{[n]} - \frac{c_1^{[n]}}{c_0^{[n]}} f_0^{[n]}\right) = \frac{1}{z}\left(f_1^{[n]} + a^{[n]} f_0^{[n]}\right) \\ &= \sum_{k=0}^{\infty} c_{0,k}^{[n+1]} z^k = c_0^{[n+1]} + O(z). \end{aligned} \tag{1.17}$$

Setting $a^{[n+1]} := -c_1^{[n+1]}/c_0^{[n+1]}$, we find that (see (1.13))

$$Q_j^{[n+1]}(z) = a^{[n+1]} Q_j^{[n]}(z) + z Q_j^{[n-1]}(z), \quad j = 1, 2, \quad n = 1, 2, \dots .$$

2. The case $m = 2$: a tuple of series $[f_0, f_1, f_2]$

2.1. **Introduction.** We are given three formal power series

$$f_0 = f_0(z) = \sum_{k=0}^{\infty} c_{0,k} z^k, \ f_1 = f_1(z) = \sum_{k=0}^{\infty} c_{1,k} z^k, \ f_2 = f_2(z) = \sum_{k=0}^{\infty} c_{2,k} z^k.$$

If we define $f_0 = c_0 + O(z)$, $f_1 = c_1 + O(z)$, $f_2 = c_2 + O(z)$, then we get

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_2 \\ f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_2 \\ f_1 \end{pmatrix}, \tag{2.1}$$

$$\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & -\dfrac{c_2}{c_1} \\ -\dfrac{c_1}{c_0} & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_2 \\ f_1 \end{pmatrix} = \begin{pmatrix} z f_0 \\ f_2 - \dfrac{c_2}{c_1} f_1 \\ f_1 - \dfrac{c_1}{c_0} f_0 \end{pmatrix} = z \begin{pmatrix} f_2^{[1]} \\ f_1^{[1]} \\ f_0^{[1]} \end{pmatrix}, \tag{2.2}$$

where

$$f_2^{[1]} := f_0, \quad f_1^{[1]} := \frac{1}{z}\left(f_2 - \frac{c_2}{c_1} f_1\right), \quad f_0^{[1]} := \frac{1}{z}\left(f_1 - \frac{c_1}{c_0} f_0\right).$$

We have

$$f_0^{[1]} = \sum_{k=0}^{\infty} c_{0,k}^{[1]} z^k = c_0^{[1]} + O(z),$$
$$f_1^{[1]} = \sum_{k=0}^{\infty} c_{1,k}^{[1]} z^k = c_1^{[1]} + O(z),$$
$$f_2^{[1]} = \sum_{k=0}^{\infty} c_{2,k}^{[1]} z^k = c_2^{[1]} + O(z).$$

Let

$$M_1 := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & -c_2/c_1 \\ -c_1/c_0 & 0 & 1 \end{pmatrix},$$
$$M^{[0]} := M_2 M_1 = \begin{pmatrix} 0 & 0 & z \\ 1 & -c_2/c_1 & 0 \\ 0 & 1 & -c_1/c_0 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix} \tag{2.3}$$

Hence relations (1.2) assume the form

$$M^{[0]}\, {}^{\mathrm{T}}\vec{f} = z\, {}^{\mathrm{T}}\vec{f}^{[1]}, \tag{2.4}$$

where $\vec{f} := (f_2, f_1, f_0)$, $\vec{f}^{[1]} := (f_2^{[1]}, f_1^{[1]}, f_0^{[1]})$, and ${}^{\mathrm{T}}\vec{v}$ is the transposition with respect to the vector row $\vec{v} = (v_1, v_2, v_3)$.

We set $f_0^{[0]} := f_0 = \sum\limits_{k=0}^{\infty} c_{0,k}^{[0]} z^k = c_0^{[0]} + O(z)$, $f_1^{[0]} := f_1 = \sum\limits_{k=0}^{\infty} c_{1,k}^{[0]} z^k = c_1^{[0]} + O(z)$, $f_2^{[0]} := f_2 = \sum\limits_{k=0}^{\infty} c_{2,k}^{[0]} z^k = c_2^{[0]} + O(z)$, $a^{[0]} := -c_2^{[0]}/c_1^{[0]}$, $b^{[0]} := -c_1^{[0]}/c_0^{[0]}$.

2.2. **Theoretical results.** Given $n \geqslant 1$, we define

$$f_2^{[n+1]} := f_0^{[n]} =: \sum_{k=0}^{\infty} c_{2,k}^{[n+1]} z^k = c_2^{[n+1]} + O(z),$$
$$f_1^{[n+1]} := \frac{1}{z}\left(f_2^{[n]} - \frac{c_2^{[n]}}{c_1^{[n]}} f_1^{[n]}\right) =: \sum_{k=0}^{\infty} c_{1,k} z^k = c_1^{[n+1]} + O(z), \tag{2.5}$$
$$f_0^{[n+1]} := \frac{1}{z}\left(f_1^{[n]} - \frac{c_1^{[n]}}{c_0^{[n]}} f_0^{[n]}\right) =: \sum_{k=0}^{\infty} c_{0,k} z^k = c_0^{[n+1]} + O(z).$$

As in (2.3), we put

$$M^{[n]} := \begin{pmatrix} 0 & 0 & z \\ 1 & -c_2^{[n]}/c_1^{[n]} & 0 \\ 0 & 1 & -c_1^{[n]}/c_0^{[n]} \end{pmatrix}. \tag{2.6}$$

Let

$$A^{[n]} := M^{[n]} \cdots M^{[0]} = \begin{pmatrix} P_1^{[n]} & P_2^{[n]} & P_3^{[n]} \\ Q_1^{[n]} & Q_2^{[n]} & Q_3^{[n]} \\ R_1^{[n]} & R_2^{[n]} & R_3^{[n]} \end{pmatrix}.$$

Then

$$\begin{aligned} A^{[n+1]} :&= \begin{pmatrix} P_1^{[n+1]} & P_2^{[n+1]} & P_3^{[n+1]} \\ Q_1^{[n+1]} & Q_2^{[n+1]} & Q_3^{[n+1]} \\ R_1^{[n+1]} & R_2^{[n+1]} & R_3^{[n+1]} \end{pmatrix} = M^{[n+1]} A^{[n]} \\ &= \begin{pmatrix} 0 & 0 & z \\ 1 & -c_2^{[n+1]}/c_1^{[n+1]} & 0 \\ 0 & 1 & -c_1^{[n+1]}/c_0^{[n+1]} \end{pmatrix} \begin{pmatrix} P_1^{[n]} & P_2^{[n]} & P_3^{[n]} \\ Q_1^{[n]} & Q_2^{[n]} & Q_3^{[n]} \\ R_1^{[n]} & R_2^{[n]} & R_3^{[n]}. \end{pmatrix}. \end{aligned} \tag{2.7}$$

From (2.7) we get the recurrence relations for $j = 1, 2, 3$ and for $n \geqslant 1$

$$\begin{aligned} P_j^{[n+1]} &= z R_j^{[n]}, \\ Q_j^{[n+1]} &= P_j^{[n]} - \frac{c_2^{[n+1]}}{c_1^{[n+1]}} Q_j^{[n]} = a^{[n+1]} Q_j^{[n]} + P_j^{[n]}, \\ R_j^{[n+1]} &= Q_j^{[n]} - \frac{c_1^{[n+1]}}{c_0^{[n+1]}} R_j^{[n]} = b^{[n+1]} R_j^{[n]} + Q_j^{[n]}, \end{aligned} \tag{2.8}$$

where

$$a^{[n+1]} := -\frac{c_2^{[n+1]}}{c_1^{[n+1]}}, \quad b^{[n+1]} := -\frac{c_1^{[n+1]}}{c_0^{[n+1]}}.$$

From (2.8) we finally get

$$\begin{aligned} Q_j^{[n+1]}(z) &= a^{[n+1]} Q_j^{[n]}(z) + z R^{[n-1]}(z), \\ R_j^{[n+1]}(z) &= b^{[n+1]} R_j^{[n]}(z) + Q_j^{[n]} \end{aligned} \tag{2.9}$$

for $j = 1, 2, 3$ and for $n = 1, 2, \ldots$. Moreover, the initial conditions for the three-term recurrence relations (2.9) follow from the equality $A^{[1]} = M^{[1]} M^{[0]}$ and read as

$$\begin{aligned} Q_1^{[0]} = 1, \quad Q_2^{[0]} = a^{[0]}, \quad Q_3^{[0]} = 0, \\ R_1^{[0]} = 0, \quad R_2^{[0]} = 1, \quad R_3^{[0]} = b^{[0]}, \\ Q_1^{[1]} = a^{[1]}, \quad Q_2^{[1]} = a^{[1]} a^{[0]}, \quad Q_3^{[1]} = z, \\ R_1^{[1]} = 1, \quad R_2^{[1]} = b^{[1]} + a^{[0]}, \quad R_3^{[1]} = b^{[1]} b^{[0]}. \end{aligned} \tag{2.10}$$

Furthermore, from (2.6) we have

$$A^{[n]} \begin{pmatrix} f_2 \\ f_1 \\ f_0 \end{pmatrix} = M^{[n]} \cdots M^{[0]} \begin{pmatrix} f_2 \\ f_1 \\ f_0 \end{pmatrix} = z^{n+1} \begin{pmatrix} f_2^{[n+1]} \\ f_1^{[n+1]} \\ f_0^{[n+1]} \end{pmatrix} = O(z^{n+1}). \tag{2.11}$$

The following result holds.

Theorem 2. *Let $k = 0, 1, 2, \dots$.*

1) If $n = 3k$, then $\deg Q_1^{[n]} = k$, $\deg Q_2^{[n]} = k$, $\deg Q_3^{[n]} = k$.

2) If $n = 3k + 1$, then $\deg R_1^{[n]} = k$, $\deg R_2^{[n]} = k$, $\deg R_3^{[n]} = k$, $\deg Q_1^{[n]} = k$, $\deg Q_2^{[n]} = k$, $\deg Q_3^{[n]} = k + 1$.

3) If $n = 3k+2$, then $\deg R_1^{[n]} = k$, $\deg R_2^{[n]} = k$, $\deg R_3^{[n]} = k+1$, $\deg Q_1^{[n]} = k$, $\deg Q_2^{[n]} = k + 1$, $\deg Q_3^{[n]} = k + 1$.

4) If $n = 3k + 3$, then $\deg R_1^{[n]} = k$, $\deg R_2^{[n]} = k + 1$, $\deg R_3^{[n]} = k + 1$.

From (2.11), we obtain

$$(R_1^{[n]}, R_2^{[n]}, R_3^{[n]})\,\mathrm{V}(f_2, f_1, f_0) = R_1^{[n]} f_2 + R_2^{[n]} f_1 + R_3^{[n]} f_0 = O(z^{n+1}). \tag{2.12}$$

The next result follows from Theorem 2 and (2.12).

Corollary 1. *Under the hypotheses of Theorem 2, for an arbitrary $n \in \mathbb{N}$, let $k_2 = \deg R_1^{[n]}$, $k_1 = \deg R_2^{[n]}$, $k_0 = \deg R_3^{[n]}$, and $\mathbf{k}^{[n]} = (k_0, k_1, k_2) \in \mathbb{Z}_+^3$ (a multiindex). Then $|\mathbf{k}^{[n]}| = k_0 + k_1 + k_2 = n - 1$, and by (2.12), the tuple $[R_3^{[n]}, R_2^{[n]}, R_1^{[n]}] = [Q_{\mathbf{k}^{[n]},0}, Q_{\mathbf{k}^{[n]},1}, Q_{\mathbf{k}^{[n]},2}]$ is a tuple of Hermite–Padé polynomials of type I for the tuple of functions $[f_0, f_1, f_2]$ and the multiindex $\mathbf{k}^{[n]}$, the order of tangency being $O(z^{|\mathbf{k}^{[n]}|+2}) = O(z^{n+1})$.*

Proof. The proof of Theorem 2 is by induction on k.

I) Let $k = 0$.

Assertions 1) and 2) of the theorem are direct consequences of representations (2.10).

Using (2.9), we find that $Q_j^{[2]} = a^{[2]} Q_j^{[1]} + z R_j^{[0]}$, $j = 1, 2, 3$. By (2.10), we have $R_1^{[0]} = 0$, $R_j^{[0]} = \mathrm{const}_j \neq 0$, $j = 2, 3$. Therefore, from (2.10) we have $\deg Q_1^{[2]} = 0$, $\deg Q_2^{[2]} = \deg Q_3^{[2]} = 1$. Similarly, an appeal to (2.9) shows that $R_j^{[2]} = b^{[2]} R_j^{[1]} + Q_j^{[1]}$. Therefore, by (2.10) we get $\deg R_1^{[2]} = \deg R_2^{[2]} = 0$, $\deg R_3^{[2]} = 1$, proving assertion 3).

Let us verify assertion 4). From (2.9), we conclude $R_j^{[3]} = b^{[3]} R_j^{[2]} + Q_j^{[2]}$. Hence by the above properties of the polynomials $R_j^{[2]}$ and $Q_j^{[2]}$, we find that $\deg R_1^{[3]} = 0$, $\deg R_2^{[3]} = \deg R_3^{[3]} = 1$.

This verifies all assertions 1)–4) of Theorem 2 for $k = 0$.

II) Now, assuming that assertions 1)–4) of Theorem 2 hold for $n = 3k$, $n = 3k + 1$, $n = 3k + 2$, and $n = 3k + 3$, respectively, let us verify them under this hypothesis with k replaced by $k+1$; that is, for $n = 3k+3$, $n = 3k+4$, $n = 3k+5$, and $n = 3k + 6$.

1) Let $n = 3(k + 1) = 3k + 3$. Then using (2.9) and since assertions 2) and 3) of Theorem 2 hold for $n = 3k + 1$ and $n = 3k + 2$, we have

$$\deg Q_1^{[3k+3]} = \max\{\deg Q_1^{[3k+2]}, 1 + \deg R_1^{[3k+1]}\} = \max\{k, 1 + k\} = k + 1,$$
$$\deg Q_j^{[3k+3]} = \max\{\deg Q_j^{[3k+2]}, 1 + \deg R_j^{[3k+1]}\} = \max\{k + 1, 1 + k\} = k + 1,$$

where $j = 2, 3$.

2) Let $n = 3k+4$. Then using (2.9) and employing assertion 4) for $n = 3k+3$ and assertion 1) for $n = 3k+3$, we have, for $j = 1, 2, 3$,

$$\deg R_j^{[3k+4]} = \max\{\deg R_j^{[3k+3]}, \deg Q_j^{[3k+3]}\} = k+1.$$

Similarly, using (2.9) and assertions 1) and 3) of Theorem 2 for $n = 3k+3$ and $n = 3k+2$, we have, respectively,

$$\begin{aligned}
&\deg Q_j^{[3k+4]} = \max\{\deg Q_j^{[3k+3]}, 1 + \deg R_j^{[3k+2]}\} = k+1 \quad \text{for} \quad j = 1, 2,\\
&\deg Q_3^{[3k+4]} = \max\{\deg Q_3^{[3k+3]}, 1 + \deg R_3^{[3k+2]}\} = k+2.
\end{aligned}$$

3) Let us verify assertion 3) for $n = 3k+5$. From (2.9) and assertion 2) for $n = 3k+4$ we have, for $j = 1, 2$,

$$\begin{aligned}
&\deg R_j^{[3k+5]} = \max\{\deg R_j^{[3k+4]}, \deg Q_j^{[3k+4]}\} = \max\{k+1, k+1\} = k+1,\\
&\deg R_3^{[3k+5]} = \max\{\deg R_3^{[3k+4]}, \deg Q_3^{[3k+4]}\} = \max\{k+1, k+2\} = k+2.
\end{aligned}$$

Similarly, using (2.9) and employing the already proved assertions 1), 2) and 4) of Theorem 2 for $n = 3k+3$, $n = 3k+4$ and $n = 3k+3$, respectively, we get

$$\begin{aligned}
&\deg Q_1^{[3k+5]} = \max\{\deg Q_1^{[3k+4]}, 1 + \deg R_1^{[3k+3]}\} = \max\{k+1, 1+k\} = k+1,\\
&\deg Q_j^{[3k+5]} = \max\{\deg Q_j^{[3k+4]}, 1 + \deg R_j^{[3k+3]}\} = k+2 \quad \text{for} \quad j = 2, 3.
\end{aligned}$$

4) Let us prove assertion 4) for $n = 3(k+1)+3 = 3k+6$. From (2.9) and since assertion 3) holds for $n = 3k+5$, we find that

$$\begin{aligned}
&\deg R_1^{[3k+6]} = \max\{\deg R_1^{[3k+5]}, \deg Q_1^{[3k+5]}\} = k+1,\\
&\deg R_j^{[3k+6]} = \max\{\deg R_j^{[3k+5]}, \deg Q_j^{[3k+5]}\} = k+2 \quad \text{for} \quad j = 2, 3.
\end{aligned}$$

Theorem 2 is proved. □

2.3. **The algorithm for $m = 2$ (a tuple of series $[f_0, f_1, f_2]$).** We are given three series: $f_0 = f_0(z) = \sum\limits_{k=0}^{\infty} c_{0,k} z^k = c_0 + O(z)$, $f_1 = f_1(z) = \sum\limits_{k=0}^{\infty} c_{1,k} z^k = c_1 + O(z)$, and $f_2 = f_2(z) = \sum\limits_{k=0}^{\infty} c_{2,k} z^k = c_2 + O(z)$.

2.3.1. **The initial iteration step: $n = 0$.** We set $f_0^{[0]} := f_0$, $f_1^{[0]} := f_1$, $f_2^{[0]} := f_2$, $c_j^{[0]} := c_j$, $j = 0, 1, 2$, $a^{[0]} := -c_2^{[0]}/c_1^{[0]}$, $b^{[0]} := -c_1^{[0]}/c_0^{[0]}$, and define

$$\begin{aligned}
&Q_1^{[0]} := 1, \quad Q_2^{[0]} := a^{[0]}, \quad Q_3^{[0]} := 0,\\
&R_1^{[0]} := 0, \quad R_2^{[0]} := 1, \quad R_3^{[0]} := b^{[0]}.
\end{aligned} \tag{2.13}$$

2.3.2. **First iteration step:** $n = 1$**.** We set

$$f_2^{[1]} := f_0^{[0]} =: \sum_{k=0}^{\infty} c_{2,k}^{[1]} z^k = c_2^{[1]} + O(z),$$
$$f_1^{[1]} := \frac{1}{z}\left(f_2^{[0]} - \frac{c_2^{[0]}}{c_1^{[0]}} f_1^{[0]}\right) = \sum_{k=0}^{\infty} c_{1,k}^{[1]} z^k = c_1^{[1]} + O(z),$$
$$f_0^{[1]} := \frac{1}{z}\left(f_1^{[0]} - \frac{c_1^{[0]}}{c_0^{[0]}} f_0^{[0]}\right) = \sum_{k=0}^{\infty} c_{0,k}^{[1]} z^k = c_0^{[1]} + O(z).$$

Given $a^{[1]} := -c_2^{[1]}/c_1^{[1]}$, $b^{[1]} := -c_1^{[1]}/c_0^{[1]}$, we put

$$Q_1^{[1]} := a^{[1]}, \quad Q_2^{[1]} := a^{[1]}a^{[0]}, \quad Q_3^{[1]} := z,$$
$$R_1^{[1]} := 1, \quad R_2^{[1]} := b^{[1]} + a^{[0]}, \quad R_3^{[1]} := b^{[1]}b^{[0]}.$$

2.3.3. **Step** $(n+1)$ **of the iteration** $(n \geqslant 1)$**.** Given $n \geqslant 1$, we put

$$f_0^{[n+1]} := \frac{1}{z}\left(f_1^{[n]} - \frac{c_1^{[n]}}{c_0^{[n]}} f_0^{[n]}\right) = \sum_{k=0}^{\infty} c_{0,k}^{[n+1]} z^k = c_0^{[n+1]} + O(z),$$
$$f_1^{[n+1]} := \frac{1}{z}\left(f_2^{[n]} - \frac{c_2^{[n]}}{c_1^{[n]}} f_1^{[n]}\right) = \sum_{k=0}^{\infty} c_{1,k}^{[n+1]} z^k = c_1^{[n+1]} + O(z),$$
$$f_2^{[n+1]} := f_0^{[n]} =: \sum_{k=0}^{\infty} c_{2,k}^{[n+1]} z^k = c_2^{[n+1]} + O(z),$$
$$a^{[n+1]} := -\frac{c_2^{[n+1]}}{c_1^{[n+1]}}, \quad b^{[n+1]} := -\frac{c_1^{[n+1]}}{c_0^{[n+1]}},$$
$$Q_j^{[n+1]}(z) = a^{[n+1]} Q_j^{[n]}(z) + z R_j^{[n-1]}(z),$$
$$R_j^{[n+1]}(z) = b^{[n+1]} R_j^{[n]}(z) + Q_j^{[n]}(z), \quad j = 1, 2, 3. \tag{2.14}$$

3. The case of an arbitrary $m \in \mathbb{N}$: a tuple of series $[f_0, \dots, f_m]$

3.1. **Introduction and theoretical results.** Given $(m+1)$ formal series $f_j = f_j(z) = \sum\limits_{j=1}^{\infty} c_{j,k} z^k$, $f_j \in \mathbb{C}[[z]]$, $j = 0, \dots, m$, we define $f_j = c_j + O(z)$. We have

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} f_m \\ f_{m-1} \\ f_{m-2} \\ \vdots \\ f_0 \end{pmatrix} = \begin{pmatrix} f_0 \\ f_m \\ f_{m-1} \\ \vdots \\ f_1 \end{pmatrix}, \tag{3.1}$$

$$\begin{pmatrix} z & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -\frac{c_m}{c_{m-1}} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{c_{m-1}}{c_{m-2}} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -\frac{c_2}{c_1} \\ -\frac{c_1}{c_0} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_m \\ f_{m-1} \\ \vdots \\ f_1 \end{pmatrix} = \begin{pmatrix} zf_0 \\ f_m - \frac{c_m}{c_{m-1}} f_{m-1} \\ \vdots \\ f_2 - \frac{c_1}{c_0} f_1 \\ f_1 - \frac{c_1}{c_0} f_0 \end{pmatrix}$$

$$= z \begin{pmatrix} f_m^{[1]} \\ f_{m-1}^{[1]} \\ f_{m-2}^{[1]} \\ \vdots \\ f_0^{[1]} \end{pmatrix} = O(z), \tag{3.2}$$

where

$$f_m^{[1]} := f_0 =: \sum_{k=0}^{\infty} c_{m,k}^{[1]} z^k = c_m^{[1]} + O(z),$$
$$f_j^{[1]} := \frac{1}{z}\left(f_{j+1} - \frac{c_{j+1}}{c_j} f_j\right) = \sum_{k=0}^{\infty} c_{j,k}^{[1]} z^k = c_j^{[1]} + O(z), \quad j = 0, \dots, m-1.$$

We set $c_j^{[0]} := c_j$,

$$f_j^{[0]} := f_j =: \sum_{k=0}^{\infty} c_{j,k}^{[0]} z^k = c_j^{[0]} + O(z), \quad j = 0, \dots, m,$$

$a_j^{[0]} := -c_j^{[0]}/c_{j-1}^{[0]}$, $j = 1, \dots, m$, and define

$$M_1 := \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$M_2 := \begin{pmatrix} z & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & a_m^{[0]} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & a_{m-1}^{[0]} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_2^{[0]} \\ a_1^{[0]} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

We also put

$$M^{[0]} := M_2 M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & z \\ 1 & a_m^{[0]} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & a_{m-1}^{[0]} & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & a_2^{[0]} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_1^{[0]} \end{pmatrix}. \tag{3.3}$$

We have

$$M^{[0]}\,{}^{\mathrm{T}}\vec{f}^{[0]} = z\,{}^{\mathrm{T}}\vec{f}^{[1]} = O(z), \text{ where } \vec{f}^{[0]} := (f_0^{[0]}, \dots, f_m^{[0]}), \quad \vec{f}^{[1]} := (f_0^{[1]}, \dots, f_m^{[1]}). \tag{3.4}$$

For $n \geqslant 1$, we define

$$a_j^{[n]} := -c_j^{[n]}/c_{j-1}^{[n]}, \quad j = 1, \dots, m,$$

$$f_m^{[n+1]} := f_0^{[n]} =: \sum_{k=0}^{\infty} c_{m,k}^{[n+1]} z^k = c_m^{[n+1]} + O(z),$$

$$f_j^{[n+1]} := \frac{1}{z}\left(f_{j+1}^{[n]} - \frac{c_{j+1}^{[n]}}{c_j^{[n]}} f_j^{[n]} \right) = \sum_{k=0}^{\infty} c_{j,k}^{[n+1]} z^k = c_j^{[n+1]} + O(z), \ j = 0, \dots, m-1. \tag{3.5}$$

Let

$$M^{[n]} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & z \\ 1 & a_m^{[n]} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & a_{m-1}^{[n]} & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & a_2^{[n]} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & a_1^{[n]} \end{pmatrix}. \tag{3.6}$$

Then, as in (3.4), we have, for $n \geqslant 1$,

$$M^{[n]}\,{}^{\mathrm{T}}\vec{f}^{[n]} = z\,{}^{\mathrm{T}}\vec{f}^{[n+1]}, \quad \text{where} \quad \vec{f}^{[n]} := (f_m^{[0]}, \dots, f_0^{[0]}) \in \mathbb{C}^{m+1}[[z]]. \tag{3.7}$$

From (3.7) it now follows that

$$M^{[n]}\cdots M^{[0]}\,{}^{\mathrm{T}}\vec{f}^{[0]} = z^{n+1}\,{}^{\mathrm{T}}\vec{f}^{[n+1]} = O(z^{n+1}),$$

and so

$$A^{[n]}\,{}^{\mathrm{T}}\vec{f}^{[0]} = z^{n+1}\,{}^{\mathrm{T}}\vec{f}^{[n+1]} = O(z^{n+1}), \tag{3.8}$$

where $A^{[n]} := M^{[n]}\cdots M^{[0]}$. Moreover, $A^{[0]} = M^{[0]}$, where the matrix $M^{[0]}$ is given by (3.3). Let

$$A^{[n]} = \begin{pmatrix} \vec{A}_1^{[n]} \\ \vdots \\ \vec{A}_{m+1}^{[n]} \end{pmatrix}, \quad \text{where} \quad \vec{A}_j^{[n]} := (A_{j,1}^{[n]},\dots,A_{j,m+1}^{[n]}), \quad j = 1,\dots,m+1. \tag{3.9}$$

By definition of the matrix $A^{[n]}$,

$$A^{[n+1]} = M^{[n+1]}A^{[n]}.$$

Therefore, an appeal to (3.6) yields

$$\vec{A}_1^{[n+1]} = z\vec{A}_{m+1}^{[n]}, \tag{3.10}$$

$$\vec{A}_j^{[n+1]} = \vec{A}_{j-1}^{[n]} + a_{m+2-j}^{[n+1]}\vec{A}_j^{[n]} = a_{m+2-j}^{[n+1]}\vec{A}_j^{[n]} + \vec{A}_{j-1}^{[n]}, \quad j = 2,\dots,m+1. \tag{3.11}$$

Now using (3.10), for $n \geqslant 1$, we transform (3.11) to read

$$\begin{aligned} \vec{A}_2^{[n+1]} &= a_m^{[n+1]}\vec{A}_2^{[n]} + z\vec{A}_{m+1}^{[n-1]}, \\ \vec{A}_j^{[n+1]} &= a_{m+2-j}^{[n+1]}\vec{A}_j^{[n]} + \vec{A}_{j-1}^{[n]}, \quad j = 3,\dots,m+1. \end{aligned} \tag{3.12}$$

Relations (3.12) are three-term recurrence relations for m rows $\vec{A}_j^{[n+1]}$, $j = 2,\dots,m+1$, of the matrix $A^{[n+1]}$. The first row is obtained from (3.10). From (3.12) it follows that in order to recursively find the entries of the rows $\vec{A}_j^{[n+1]}$, $j = 2,\dots,m+1$, of the matrix $A^{[n+1]}$ it suffices to know the entries of the rows $\vec{A}_j^{[1]}$, $j = 2,\dots,m+1$, of the matrix $A^{[1]}$ and the $(m+1)$st row $\vec{A}_{m+1}^{[0]}$ of the original matrix $A^{[0]}$. Let us find the required rows.

From (3.3) and since $A^{[0]} = M^{[0]}$ we get

$$\vec{A}_{m+1}^{[0]} = (\underbrace{0,0,0,0,\dots,0,1,a_1^{[0]}}_{m+1}) \in \mathbb{C}^{m+1}.$$

Next, since $A^{[1]} := M^{[1]}M^{[0]}$ and using (3.6) (for $n = 1$), we obtain

$$\begin{aligned} \vec{A}_2^{[1]} &= (a_m^{[1]}, a_m^{[1]}a_m^{[0]}, 0, 0, \dots, 0, 0, z) \in \mathbb{C}^{m+1}, \\ \vec{A}_j^{[1]} &= (0,\dots,0,\underbrace{1, a_{m+2-j}^{[1]} + a_{m+3-j}^{[0]}, a_{m+2-j}^{[1]}a_{m+2-j}^{[0]}}_{j-2\,,j-1\,,j}, 0,\dots,0), \ j = 3,\dots,m+1, \\ \vec{A}_{m+1}^{[1]} &= (0,0,\dots,0,1,a_1^{[1]} + a_2^{[0]}, a_1^{[1]}a_1^{[0]}) \in \mathbb{C}^{m+1}. \end{aligned} \tag{3.13}$$

So, the initial conditions for the recurrence relations (3.12) are given by $(m+1)$ vectors from the space $\mathbb{C}^{m+1}$. By (3.12), all entries $A_{j,k}^{[n]}$ are polynomials of z, $A_{j,k}^{[n]} \in \mathbb{C}[z]$, for all $n = 0, 1, \dots$, and $j, k = 1,\dots,m+1$.

Given an arbitrary vector $\vec{c} = (c_1, \dots, c_{m+1}) \in \mathbb{C}^{m+1}$, we set

$$ {}^{\mathrm{B}}\vec{c} := (c_{m+1}, \dots, c_1). $$

As in (3.12), we have

$$ A^{[n+1]}_{2,k} = a^{[n+1]}_n A^{[n]}_{2,k} + z A^{[n-1]}_{m+1,k}, \quad k = 1, \dots, m+1, $$
$$ A^{[n+1]}_{j,k} = a^{[n+1]}_{m+2-j} A^{[n]}_{j,k} + A^{[n]}_{j-1,k}, \quad k = 1, \dots, m+1, \quad j = 3, \dots, m+1. $$

The following result holds.

Theorem 3. *Let $m \in \mathbb{N}$, $n \geqslant m-1$ and $n = m - 1 + (m+1)k + \ell$, where $\ell \in \{0, \dots, m\}$, $k \in \{0, 1, 2, \dots\}$ ($\ell = (n - m + 1) \pmod{m+1}$). Then*

1) If $\ell = 0$, then $\deg A^{[n]}_{m+1,s} = k$ for all $s = 1, \dots, m+1$. The vector ${}^{\mathrm{B}}\vec{A}^{[n]}_{m+1} = \vec{Q}_{\mathbf{k}^{[n]}}(\vec{f}) = (Q_{\mathbf{k}^{[n]},0}, Q_{\mathbf{k}^{[n]},1}, \dots, Q_{\mathbf{k}^{[n]},m})$ is the vector of Hermite–Padé polynomials of type I for the vector series ${}^{\mathrm{B}}\vec{f} = (f_0, \dots, f_m)$ and the multiindex $\mathbf{k}^{[n]} = (k, k, \dots, k)$. The order of tangency is $|\mathbf{k}^{[n]}| + m = (m+1)k + m = n + 1$; that is,

$$ {}^{\mathrm{B}}\vec{Q}_{\mathbf{k}^{[n]}}(\vec{f})\, {}^{\mathrm{T}}\vec{f} = \vec{A}^{[n]}_{m+1}\, {}^{\mathrm{T}}\vec{f} = O(z^{n+1}). \tag{3.14} $$

2) If $\ell = 1, \dots, m$, then $\deg A^{[n]}_{m+1,m+1-s} = k+1$ for $s = 0, \dots, \ell - 1$ and $\deg A^{[n]}_{m+1,m+1-s} = k$ for $s = \ell, \dots, m$. Moreover, the vector ${}^{\mathrm{B}}\vec{A}^{[n]}_{m+1} = \vec{Q}_{\mathbf{k}^{[n]}}(\vec{f}) = (Q_{\mathbf{k}^{[n]},0}, Q_{\mathbf{k}^{[n]},1}, \dots, Q_{\mathbf{k}^{[n]},m})$ is the vector of Hermite–Padé polynomials of type I for the vector function ${}^{\mathrm{B}}\vec{f} = (f_0, \dots, f_m)$ and the multiindex

$$ \mathbf{k}^{[n]} = (\underbrace{k+1, \dots, k+1}_{\ell}, \underbrace{k, \dots, k}_{m+1-\ell}). $$

The order of tangency is $|\mathbf{k}^{[n]}| + m = (m+1)k + m + \ell = n + 1$; that is,

$$ {}^{\mathrm{B}}\vec{Q}_{\mathbf{k}^{[n]}}(\vec{f})\, {}^{\mathrm{T}}\vec{f} = \vec{A}^{[n]}_{m+1}\, {}^{\mathrm{T}}\vec{f} = O(z^{n+1}). \tag{3.15} $$

So, for any $n \geqslant m-1$, $Q_{\mathbf{k}^{[n]},j} = A^{[n]}_{m+1,m+1-j}$, $j = 0, \dots, m$, are Hermite–Padé polynomials of type I for the tuple $[f_0, \dots, f_m]$ and the multiindex $\mathbf{k}^{[n]} = (k_0, \dots, k_m)$, where $k_j = \deg A^{[n]}_{m+1,m+1-j}$, $j = 0, \dots, m$, and the order of tangency is

$$ |\mathbf{k}^{[n]}| + m = \sum_{j=0}^{m} \deg A^{[n]}_{m+1,m+1-j} + m = n + 1 $$

(the multiindex $\mathbf{k}^{[n]}$ is uniquely determined from the given number $n \geqslant m-1$).

Remark 3. Note that in the above recurrence algorithm all tuples of Hermite–Padé polynomials corresponding to the multiindices of the form $(k, k, k, \dots, k, k)$, $(k+1, k, k, \dots, k, k)$, $(k+1, k+1, k, \dots, k, k), \dots, (k+1, k+1, k+1, \dots, k+1, k)$ are already evaluated by the time when the tuple of Hermite–Padé polynomials corresponding to the multiindex $(k+1, k+1, k+1, \dots, k+1, k+1)$ is found. According to [15], such indices are called *proper*.

Remark 4 ((cf. Remark 2))**.** If the iteration process is started from the vector function $\vec{f}^{[0]} = (f_0, \dots, f_m) \in \mathbb{C}^{m+1}$ (instead of $\vec{f}^{[0]} = (f_m, \dots, f_0)$), then we get Hermite–Padé polynomials of type I for the multiindices $(k, \dots, k, k, k), \dots,$ $(k, \dots, k, k, k+1)$, $(k, \dots, k, k+1, k+1), \dots$.

If we set $\vec{f}^{[0]} = (\underbrace{f_{s-1}, \dots, f_0}_{s}, f_m, \dots, f_s)$, then the iteration process will produce Hermite–Padé polynomials for the multiindices (cf. [14]):

$$(k, \dots, k), \ (k, \dots, k, \underbrace{k+1}_{s}, k, k, \dots, k), \ (k, \dots, k, \underbrace{k+1, k+1}_{s,s+1}, k, \dots, k).$$

Remark 5. Hermite–Padé polynomials are known to satisfy many different recurrence relations, which are widely useful in theoretical studies of Hermite–Padé polynomials (see, for example, [12], [13], [3] and the references given therein). However, as a rule, such results are obtained for a system of Laurent series defined at the point at infinity and are, in general, more involved than the three-term relations (3.12).

Proof of Theorem 3. For vector $\vec{p}(z) := (p_1(z), \dots, p_{m+1}(z)) \in \mathbb{C}^{m+1}[z]$, where $p_j(z) \in \mathbb{C}[z]$, we define

$$\deg \vec{p}(z) := (\deg p_1(z), \dots, \deg p_{m+1}(z)) \in \mathbb{Z}_+^{m+1}.$$

For vectors $\vec{p}(z)$ and $\vec{q}(z)$, $\vec{p}, \vec{q} \in \mathbb{C}^{m+1}[z]$, the inequality $\deg \vec{p} \geqslant \deg \vec{q}$ means by definition that $\deg \vec{p} - \deg \vec{q} \in \mathbb{Z}_+^{m+1}$. Similarly, the inequality $\deg \vec{p} > \deg \vec{q}$ means that $\deg \vec{p} - \deg \vec{q} \in \mathbb{N}^{m+1}$.

We set $\vec{e}_1 := (1, 0, \dots, 0) \in \mathbb{Z}^{m+1}$ and define $\vec{e}_{m+1} := (1, 1, \dots, 1) \in \mathbb{N}^{m+1}$.

The following Lemma 1, which holds under the hypotheses of Theorem 3, is the main ingredient in the proof of this theorem.

Lemma 1. *Let $n \geqslant m-1$. Then*

$$\deg \vec{A}_{j-1}^{[n]} \geqslant \deg \vec{A}_j^{[n]} \quad \text{for} \quad j = 2, \dots, m. \tag{3.16}$$

Let $n = m-1+(m+1)k+\ell$, where $\ell = (n-(m-1)) \pmod{m+1} \in \{0, 1, \dots, m\}$, $k = 0, 1, \dots$. Then the following relations hold:

$$\text{for all} \quad \ell \quad \deg \vec{A}_{\ell+s}^{[n]} = (\underbrace{k, \dots, k}_{s}, \underbrace{k+1, \dots, k+1}_{m+1-s}), \quad s = 1, \dots, m+1-\ell, \tag{3.17}$$

$$\text{for} \quad \ell \geqslant 1 \quad \deg \vec{A}_{\ell-j}^{[n]} = (\underbrace{k+1, \dots, k+1}_{m+1-j}, \underbrace{k+2, \dots, k+2}_{j}), \quad j = 0, \dots, \ell-1. \tag{3.18}$$

From relations (3.17) and (3.18) of Lemma 1 for $s = m+1-\ell$ and $j = \ell - 1$ for $\ell \geqslant 1$ we get, respectively, that

$$\begin{aligned} \deg \vec{A}_{m+1}^{[n]} &= (\underbrace{k, \dots, k}_{m+1-\ell}, \underbrace{k+1, \dots, k+1}_{\ell}) \quad \text{and} \\ \deg \vec{A}_1^{[n]} &= (\underbrace{k+1, \dots, k+1}_{m+2-\ell}, \underbrace{k+2, \dots, k+2}_{\ell-1}). \end{aligned} \tag{3.19}$$

Now (3.19) yields $\deg A_{m+1,j}^{[n]} = k$ for $j = 1, \dots, m+1-\ell$ and $\deg A_{m+1,j}^{[n]} = k+1$ for $j = m+1-\ell+1, \dots, m+1$. These relations can be equivalently written as $\deg A_{m+1,m+1-s}^{[n]} = k+1$ for $s = 0, \dots, \ell-1$ and $\deg A_{m+1,m+1-s}^{[n]} = k$ for $s = \ell, \dots, m$. This implies assertions 1) and 2) of Theorem 3.

Theorem 1 is proved. □

Proof of Lemma 1. We argue by induction on n.

1) Let $n = m - 1$. Then

$$A^{[n]} = M^{[n]}A^{[n-1]} = M^{[n]} \cdots M^{[1]}M^{[0]}, \tag{3.20}$$

where the matrix $M^{[0]}$ has the form (3.3), and the matrix $M^{[n]}$ has the analogous form (3.6). Let $M^{[p]}$ be the matrix obtained from the matrix $M^{[n]}$ by replacing n by p, $p = 0, 1, \ldots, n$. Then $M^{[p]} \in \mathrm{GL}_{\mathbb{C}}(m+1, m+1)$. Moreover, $\det M^{[p]} = (-1)^p z \not\equiv 0$.

Let $B_2 := M^{[p]}B_1$, where $B_1 \in \mathrm{GL}_{\mathbb{C}}(m+1, m+1)$. The matrix B_2 is the matrix B_1 multiplied on the left by the matrix $M^{[p]}$. The following facts are direct consequences of the structure (3.6) of the matrix $M^{[p]}$:

1) the last $(m+1)$st row of the matrix B_1 is multiplied by z and becomes the first row of the matrix B_2; so, the degree of the corresponding polynomial elements of the original matrix B_1 is increased precisely by 1;

2) the second row of the matrix B_2 is obtained as follows: the second row of the matrix B_1 is multiplied by the entry $a_m^{[p]}$ and is summed with an entry of the first row of the matrix B_1;

3) for $j \geqslant 2$, the corresponding jth row of the matrix B_2 is obtained as follows: the jth row of the matrix B_1 is multiplied by the entry $a_{m-j+2}^{[p]}$ and is summed with an entry of the $(j-1)$st row of the matrix B_1.

Hence, since now we now know the explicit form (3.3) of the matrix $M^{[0]}$, it follows that for $n = m - 1$ the matrix $A^{[m-1]} = M^{[m-1]} \cdots M^{[0]}$ reads as

$$\begin{pmatrix} * & *z+* & *z+* & *z+* & \ldots & *z+* & *z+* & *z+* \\ * & * & *z+* & *z+* & \ldots & *z+* & *z+* & *z+* \\ * & * & * & *z+* & \ldots & *z+* & *z+* & *z+* \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ * & * & * & * & \ldots & * & * & *z+* \\ c_m & c_{m-1} & c_{m-2} & c_{m-3} & \ldots & c_2 & c_1 & c_0 \end{pmatrix}, \tag{3.21}$$

where '$*$' and '$*z+*$' denote, respectively, some complex quantities and some first-degree polynomials of z with complex coefficients. The precise values of these quantities and the explicit form of these polynomials is immaterial here. The quantities $c_0, c_1, \ldots, c_m$ are some complex constant which are not simultaneously zero.[1] Moreover, from (3.7) and (3.8) we have

$$c_0 f_0 + c_1 f_1 + \cdots + c_{m-1} f_{m-1} + c_m f_m = O(z^m) = O(z^{n+1}). \tag{3.22}$$

Relation (3.22) means that the nontrivial tuple of constant quantities $[c_0, c_1, \ldots, c_{m-1}, c_m]$ is a tuple of Hermite–Padé polynomials of type I for the tuple of series $[f_0, f_1, \ldots, f_{m-1}, f_m]$ and the multiindex $\mathbf{k}^{[m-1]} = (0, 0, \ldots, 0, 0)$ with the corresponding order of tangency, which is equal to $|\mathbf{k}^{[m-1]}| + m = m = n + 1$.

Thus, representation (3.21) implies that the conclusions of Lemma 1 with $n = m - 1$ are true.

2) Assuming now that the conclusions of Lemma 1 hold for some $n \geqslant m - 1$, let us show that these results also hold with n replaced by $n + 1$.

Let $n = m - 1 + (m+1)k + \ell$, $\ell = (n - m + 1) \pmod{m+1} \in \{0, \ldots, m\}$.

[1] Here it is worth recalling that all the original series $f_0, \ldots, f_m$ are in general position. So, $(c_0, \ldots, c_m) \neq (0, \ldots, 0)$, inasmuch as $\det M^{[p]} = (-1)^p z \not\equiv 0$, and similarly, $\det A^{[p]} = (-1)^{pm} z \not\equiv 0$.

a) Let us prove inequality (3.16); that is, we need to show that $\deg \vec{A}_j^{[n+1]} \geqslant \deg \vec{A}_{j+1}^{[n+1]}$ for $j = 1, \ldots, m$.

Since (3.19) holds by the induction assumption, from (3.10) we have $\deg \vec{A}_1^{[m+1]} = (k+1, \ldots, k+1)$ for $\ell = 0$ and $\deg \vec{A}_1^{[m+1]} = (\underbrace{k+1, \ldots, k+1}_{m+1-\ell}, \underbrace{k+2, \ldots, k+2}_{\ell})$ for $\ell \geqslant 1$. Moreover, $\deg \vec{A}^{[n]} = (k, \underbrace{k+1, \ldots, k+1}_{m})$ for $\ell = 0$ and $\deg \vec{A}^{[n]} = (k+1, \ldots, k+1)$ for $\ell = 1$. As a result, $\deg \vec{A}_1^{[n+1]} > \deg \vec{A}_1^{[n]}$ for all ℓ. Using (3.17) and (3.18), which hold by the induction assumption, we get $\deg \vec{A}_1^{[n+1]} > \deg \vec{A}_1^{[n]}$.

Now an appeal to the recurrence relations (3.11) shows that

$$\deg \vec{A}_1^{[n+1]} \geqslant \deg \vec{A}_1^{[n]} = \deg \vec{A}_2^{[n+1]}. \tag{3.23}$$

The inequality

$$\deg \vec{A}_j^{[n+1]} \geqslant \deg \vec{A}_{j+1}^{[n+1]}, \quad j = 2, \ldots, m, \tag{3.24}$$

follows from the recurrence relation (3.11) and the induction assumption. Hence from (3.23) and (3.24) we get

$$\deg \vec{A}_j^{[n+1]} \geqslant \deg \vec{A}^{[n+1]}, \quad j = 1, 2, \ldots, m.$$

2) Let us now prove (3.17) and (3.18) for the next induction step; that is, for $n+1$. There are two cases to consider: $\ell \in \{0, \ldots, m-1\}$ and $\ell = m$.

a) Let $\ell \in \{0, \ldots, m-1\}$. Then $\ell + 1 \in \{1, \ldots, m\}$ and $n + 1 = m - 1 + (m+1)k + \ell + 1$. From (3.10) and (3.17) for $s = m + 1 - \ell$ we get

$$\begin{aligned}\deg \vec{A}_1^{[n+1]} = \deg \vec{A}_{m+1}^{[n+1]} + \vec{e}_{m+1} &= (\underbrace{k+1, \ldots, k+1}_{m+1-\ell}, \underbrace{k+2, \ldots, k+2}_{\ell}) \\ &= (\underbrace{k+1, \ldots, k+1}_{m+1-j}, \underbrace{k+2, \ldots, k+2}_{j})\big|_{j=\ell} = \deg \vec{A}_{\ell'-j}^{[n+1]}\big|_{j=\ell'-1},\end{aligned}$$

where $\ell' = \ell + 1$. This proves formula (3.18) for $n+1$ and $j = \ell = \ell' - 1$.

By the induction assumption, $\deg \vec{A}_{j-1}^{[n]} \geqslant \deg \vec{A}_j^{[n]}$ for $j = 2, \ldots, m+1$, and hence from (3.11) we have

$$\deg \vec{A}_j^{[n+1]} = \deg \vec{A}_{j-1}^{[n]}, \quad j = 2, \ldots, m+1. \tag{3.25}$$

Therefore, for $\ell \in \{1, \ldots, m-1\}$ and $\ell' = \ell + 1 \in \{2, \ldots, m\}$ it follows by (3.17) that

$$\begin{aligned}\deg \vec{A}_{\ell'+s}^{[n+1]} = \deg \vec{A}_{\ell+s}^{[n]} &= (\underbrace{k, \ldots, k}_{s}, \underbrace{k+1, \ldots, k+1}_{m+1-s}), \\ s &= 1, \ldots, m+1-\ell' = 1, \ldots, m-\ell.\end{aligned} \tag{3.26}$$

This proves (3.17) for the induction step $(n+1)$.

Let us verify (3.18). By (3.11) and using inequality (3.16), we have

$$\deg \vec{A}_{\ell'-j}^{[n+1]} = \deg \vec{A}_{\ell'-j-1}^{[n]} = \deg \vec{A}_{\ell-j}^{[n]} = (\underbrace{k+1, \ldots, k+1}_{m+1-j}, \underbrace{k+2, \ldots, k+2}_{j}), \tag{3.27}$$

where $j = 0, \dots, \ell' - 2 = 0, \dots, \ell - 1$. For $j = \ell' - 1 = \ell$, we have $\deg \vec{A}^{[n+1]}_{\ell'-j} = \deg \vec{A}^{[n+1]}_1$; (3.27) was proved above.

b) Now let us assume that $\ell = m$. Then $n + 1 = m - 1 + (m + 1)(k + 1)$ and $\ell' = 0$. In this case, from (3.10) and (3.19) we have $s = m + 1 - s$ and

$$\deg \vec{A}^{[n+1]}_1 = \vec{e}_1 + \deg \vec{A}^{[n]}_{m+1} = (k, \underbrace{k+1, \dots, k+1}_{m}) + \vec{e}_1 = (k+1, \underbrace{k+2, \dots, k+2}_{m})$$

$$= \deg \vec{A}^{[n+1]}_{\ell'+s}\big|_{\ell'=0, s=1} = (\underbrace{k+1, \dots, k+1}_{s}, \underbrace{k+2, \dots, k+2}_{m+1-s})\big|_{s=1}.$$

Next, for $s = 2, \dots, m+1$, $s - 1 = s' = m - j'$, we have

$$\deg \vec{A}^{[n+1]}_{\ell'+s} = \deg \vec{A}^{[n+1]}_s = \deg \vec{A}^{[n]}_{s-1} = \deg \vec{A}^{[n]}_{m-j'}$$
$$= (\underbrace{k+1, \dots, k+1}_{m+1-j'}, \underbrace{k+2, \dots, k+2}_{j'}) = (\underbrace{k+1, \dots, k+1}_{s}, \underbrace{k+2, \dots, k+2}_{m+1-s}).$$

Hence (3.17) is proved in the $(n+1)$st step also for $\ell = m$.

Lemma 1 is proved. □

3.2. **The algorithm for an arbitrary $m \in \mathbb{N}$ (a tuple of series $[f_0, \dots, f_m]$).**

3.2.1. **Construction of new series $f_0^{[n]}, \dots, f_m^{[n]}$, $n = 1, 2, \dots$.** We are given $m \in \mathbb{N}$ and series $f_0, \dots, f_m \in \mathbb{C}[[z]]$, $f_j = f_j(z) = \sum\limits_{k=0}^{\infty} c_{j,k} z^k = c_j + O(z)$.

Initial step. We set $f_j^{[0]} := f_j = \sum\limits_{k=0}^{\infty} c_{j,k} z^k =: \sum\limits_{k=0}^{\infty} c_{j,k}^{[0]} z^k = c_j^{[0]} + O(z)$, $j = 1, \dots, m$; $a_j^{[0]} := -c_j^{[0]}/c_{j-1}^{[0]}$, $j = 1, \dots, m$.

Step 1. We define

$$f_m^{[1]} := f_0^{[0]} =: \sum_{k=0}^{\infty} c_{m,k}^{[1]} z^k = c_m^{[1]} + O(z),$$

$$f_j^{[1]} := \frac{1}{z}\left(f_{j+1}^{[0]} - \frac{c_{j+1}^{[0]}}{c_j^{[0]}} f_j^{[0]}\right) = \sum_{k=0}^{\infty} c_{j,k}^{[1]} z^k = c_j^{[1]} + O(z), \quad j = 0, \dots, m-1,$$

$$a_j^{[1]} := -c_j^{[1]}/c_{j-1}^{[1]}, \quad j = 1, \dots, m.$$

Step $(n+1)$ **$(n \geqslant 1)$**. We set

$$f_m^{[n+1]} := f_0^{[n]} =: \sum_{k=0}^{\infty} c_{m,k}^{[n+1]} z^k = c_m^{[n+1]} + O(z),$$

$$f_j^{[n+1]} := \frac{1}{z}\left(f_{j+1}^{[n]} + a_{j+1}^{[n]} f_j^{[n]}\right) = \sum_{k=0}^{\infty} c_{j,k}^{[n+1]} z^k = c_j^{[n+1]} + O(z), \quad j = 0, \dots, m-1,$$

$$a_j^{[n+1]} := -c_j^{[n+1]}/c_{j-1}^{[n+1]}, \quad j = 1, \dots, m.$$

So, from a given tuple of series $f_0, \dots, f_m \in \mathbb{C}[[z]]$, we have constructed new series $f_0^{[n]}, \dots, f_m^{[n]} \in \mathbb{C}[[z]]$ and found the quantities $a_1^{[n]}, \dots, a_m^{[n]} \in \mathbb{C}$ for all $n = 0, 1, 2, \dots$.

Remark 6. Note that the main purpose of this step in § 3.2.1 is to find the m quantities $a_1^{[n]}, \dots, a_m^{[n]} \in \mathbb{C}$ $(n = 1, 2, \dots)$.

Remark 7. By definition of the step of § 3.2.1 the new series $f_0^{[n+1]}, \dots, f_m^{[n+1]}$ can be evaluated from the series $f_0^{[n]}, \dots, f_m^{[n]}$ in parallel.

3.2.2. **Construction of Hermite–Padé polynomials for a multiindex $\mathbf{k}^{[n]} \in \mathbb{Z}_+^{m+1}$ (see (3.28)) for $n \geqslant m-1$.**

Initial step. We set

$$\vec{A}_{m+1}^{[0]} := (0, 0, 0, \dots, 0, 1, a_1^{[0]}) \in \mathbb{C}^{m+1}.$$

Step 1. We set

$$\vec{A}_2^{[1]} := (a_m^{[1]}, a_m^{[1]} a_m^{[0]}, 0, 0, \dots, 0, 0, z) \in \mathbb{C}^{m+1},$$

for $j = 3, \dots, m+1$ we set

$$\vec{A}_j^{[1]} := (0, \dots, 0, \underbrace{1, a_{m+2-j}^{[1]} + a_{m+3-j}^{[0]}, a_{m+2-j}^{[1]} a_{m+2-j}^{[0]}}_{j-2,\ j-1,\ j}, 0, \dots, 0) \in \mathbb{C}^{m+1}.$$

Step $(n+1)$**,** $n \geqslant m-1$. Setting $\ell := (n-(m-1)) \pmod{m+1} \in \{0, \dots, m\}$, we find k from the relation $n-(m-1) = (m+1)k + \ell$ (we have $\ell = 0, \dots, m$, $k := (n-(m-1)-\ell)/(m+1) \in \mathbb{Z}_+$, $k = 0, 1, \dots$).

Next, we put

$$\mathbf{k}^{[n]} := (\underbrace{k+1, \dots, k+1}_{\ell}, \underbrace{k, \dots, k}_{m+1-\ell}) \in \mathbb{Z}_+^{m+1}, \tag{3.28}$$

$$\vec{A}_2^{[n+1]} := a_m^{[n+1]} \vec{A}_2^{[n]} + z \vec{A}_{m+1}^{[n-1]},$$

$$\vec{A}_j^{[n+1]} := a_{m+2-j}^{[n+1]} \vec{A}_j^{[n]} + \vec{A}_{j-1}^{[n]}, \quad j = 3, \dots, m+1.$$

Now, for all $n \geqslant m-1$, we have

$$\vec{A}_{m+1}^{[n]} \, {}^{\mathrm{T}}\vec{f} = O(z^{n+1})$$

and the vector $\vec{Q}_{\mathbf{k}^{[n]}}(\vec{f}) := (A_{m+1,m+1}^{[n]}, \dots, A_{m+1,1}^{[n]}) = (Q_{\mathbf{k}^{[n]},0}, \dots, Q_{\mathbf{k}^{[n]},m})$ is the vector of Hermite–Padé polynomials of type I for the vector series ${}^{\mathrm{B}}\vec{f} := (f_0, \dots, f_m)$ and the multiindex $\mathbf{k}^{[n]} = (k_0, \dots, k_m)$, where $k_j = \deg A_{m+1,m+1-j}^{[n]}$, $j = 0, \dots, m$, and the order of tangency is

$$|\mathbf{k}^{[n]}| + m = \sum_{j=0}^{m} \deg A_{m+1,m+1-j}^{[n]} + m = n+1$$

(the multiindex $\mathbf{k}^{[n]}$ is uniquely determined from the given number $n \geqslant m-1$; see (3.28)).

References

[1] Paolo Amore, John P. Boyd, Francisco M. Fernandez, "High order analysis of the limit cycle of the van der Pol oscillator", 012702; Published Online: 30 January 2018, *J. Math. Phys.*, **59**:1 (2018), 11 pp.

[2] George A. Baker, Jr., Peter Graves-Morris, *Padé approximants*, Second edition, Encyclopedia of Mathematics and its Applications, 59, Cambridge University Press, Cambridge, 1996, xiv+746 pp. ISBN: 0-521-45007-1.

[3] D. Barrios Rolanía, J. S. Geronimo, G. López Lagomasino, "High-order recurrence relations, Hermite–Padé approximation and Nikishin systems", *Sb. Math.*, **209**:3 (2018), 385–420.
[4] B. Beckermann, G. Labahn, "A uniform approach for Hermite–Padé and simultaneous Padé approximants and their matrix-type generalizations", *Numer. Algor.*, **3**, 1992, 45–54.
[5] Bernhard Beckermann, George Labahn, "Fraction-free computation of matrix rational interpolants and matrix GCDs", *SIAM J. Matrix Anal. Appl.*, **22**:1 (2000), 114–144.
[6] J. Della Dora, C. Dicrescenzo, "Approximation de Padé–Hermite", (French) [Padé–Hermite approximation], *Padé approximation and its applications* (Proc. Conf., Univ. Antwerp, Antwerp, 1979), Lecture Notes in Math., **765**, Springer, Berlin–New York, 1979, 88–115.
[7] Harm Derksen, "An algorithm to compute generalized Padé–Hermite forms", Tech. Rep. Rep. 9403, 1994, Catholic University Nijmegen (January 1994).
[8] M. Fasondini, N. Hale, R. Spoerer, J. Weideman, "Quadratic Padé approximation: numerical aspects and applications", *Computer Research and Modeling*, **11**:6 (2019), 1017–1031.
[9] T. M. Feil, H. H. H. Homeier, "Programs for the approximation of real and imaginary single- and multi-valued functions by means of Hermite–Padé-approximants", *Comput. Phys. Commun.*, **158**:2 (2004), 124–135.
[10] A. V. Komlov, N. G. Kruzhilin, R. V. Palvelev, S. P. Suetin, "Convergence of Shafer quadratic approximants", *Russian Math. Surveys*, **71**:2 (2016), 373–375.
[11] A. V. Komlov, "Polynomial Hermite–Padé m-system and reconstruction of the values of algebraic functions", Submitted, 2020 (Research Perspectives CRM Barcelona, IRP SAFAIS 2019), Trends in Mathematics, 2020, ISSN: 2509-7407.
[12] A. López-Garcia and G. López Lagomasino, "Nikishin systems on star-like sets: Ratio asymptotics of the associated multiple orthogonal polynomials", *J. Approx. Theory*, **225** (2018), 1–40.
[13] A. López-Garcia, E. Mina-Diaz, "Nikishin systems on star-like sets: algebraic properties and weak asymptotics of the associated multiple orthogonal polynomials", *Sb. Math.*, **209**:7 (2018), 1051–1088.
[14] Toshiyuki Mano, Teruhisa Tsuda, "Hermite–Padé approximation, isomonodromic deformation and hypergeometric integral", *Math. Z.*, **285**:1–2 (2017), 397–431.
[15] Nikishin, E. M.; Sorokin, V. N., *Rational approximations and orthogonality*, Translated from the Russian by Ralph P. Boas, Translations of Mathematical Monographs, 92, American Mathematical Society, Providence, RI, 1991.
[16] V. I. Parusnikov, "The Jacobi–Perron algorithm and simultaneous approximation of functions", *Math. USSR-Sb.*, **42**:2 (1982), 287–296.
[17] T. Sakurai, T. Torii, and H. Sugiura, "An iterative method for algebraic equation by Padé approximation", *Computing*, **46** (1991), 131–141.
[18] Shengfeng Li and Yi Dong, "Viscovatov-Like Algorithm of Thiele–Newton's Blending Expansion for a Bivariate Function", *Mathematics*, **2019**, 7(8), 696.
[19] A. V. Sergeev, "A recursive algorithm for Padé–Hermite approximants", *U.S.S.R. Comput. Math. Math. Phys.*, **26**:2 (1986), 17–22.
[20] Alexei V Sergeev and David Z Goodson, "Summation of asymptotic expansions of multiple-valued functions using algebraic approximants: Application to anharmonic oscillators", *Journal of Physics A: Mathematical and General*, **31**:18 (1998), 4301–4317.
[21] Sergey P. Suetin, Hermite–Padé polynomials and analytic continuation: new approach and some results, 2018, 45 pp.; arxiv: 1806.08735.
[22] A. Trias, "Method to obtain all power flow solutions in radial chain-like networks", US Patent Application. Serial No. 62/488,810, 2017, (A. Trias, System and method for monitoring and managing electrical power transmission and distribution networks. US Patents 7,519,506; 2009 and 7,979,239; 2011).
[23] A. Trias, "Sigma algebraic approximants as a diagnostic tool in power networks", US Patent 9,563,722; 2017.
[24] Antonio Trias, "HELM: The Holomorphic Embedding Load-Flow Method: Foundations and Implementations", Foundations and Trends in Electric Energy Systems, 3:3–4, 140–370.
[25] J. Van Iseghem, "Vector orthogonal relations. Vector QD-algorithm", *J. Comput. Appl. Math.*, **19**:1 (1987), 141–150.

[26] J. Van Iseghem, "Convergence of the vector QD-algorithm. Zeros of vector orthogonal polynomials", *J. Comput. Appl. Math.*, **25**:1 (1989), 33–46.

[27] B. Viscovatov [Viscovatoff], "De la methode generale pour reduire toutes sortes des quantites en fractions continues", [Trans.: A general method for reducing all types of quantities to continued fractions, submitted 18 Dec 1805], Memoires de L'Academie Imperiale des Sciences de St. Petersburg, I (Tome I), 1803–1806, 226–247.

[28] Rastko Zivanovic, "Continuation via quadratic approximation to reconstruct solution branches and locate singularities in the power flow problem", *2016 24th Mediterranean Conference on Control and Automation (MED)* (Athens), 2016, 866–870.

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
Email address: nikonomov@math.bas.bg

Steklov Mathematical Institute of the Russian Academy of Sciences
Email address: suetin@mi-ras.ru